Constructive analysis of the incompressible Navier-Stokes equation

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Abstract

A global time-discretized scheme for the Navier-Stokes equation system in its Leray projection form is defined. It is shown that the scheme converges to a bounded global classical solution for data which have polynomial decay at infinity. Furthermore, the algorithm proposed is extended to the situation of initial-boundary value problems. Algorithms constructed in a different context (cf. [4, 10, 5, 9]) may be used within the proposed scheme in order to compute the solution of Leray's form of the Navier-Stokes system. The method can be extended to incompressible Navier-Stokes equation systems on manifolds, which is shown in a subsequent paper.

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1 Introduction

In its Leray projection form the incompressible Navier-Stokes equation is a semilinear partial integro-differential equation system, where the integral term is quadratic with respect to the gradient of velocity. This integral term requires a careful treatment in order to control the growth of the solution (whatever scheme you propose). This is the main difference to the multivariate Burgers equation. Indeed, for the multivariate Burgers equation, i.e., the Cauchy problem

$$\left\lbrace\begin{array}{ll}\frac{\partial u_i}{\partial t}=\nu\sum_{j=1}^n \frac{\partial^2 u_i}{\partial x_j^2}-\sum_{j=1}^n u_j\frac{\partial u_i}{\partial x_j},\\ \\ \mathbf{u}(0,.)=\mathbf{h},\end{array}\right. \tag{1}$$

on $[0,\infty)\times\mathbb{R}^n$ (where ν is some strictly positive constant, i.e., $\nu>0$), and $1\leq i\leq n$) global solutions may be constructed via the a priori estimate

$$\max_j \sup_{x\in\Omega}|u_j(t,x)|\leq \max_j \sup_{x\in\Omega}|h_j(x)|, \tag{2}$$

which may be obtained from estimates of the form

$$\frac{\partial}{\partial t}\|u(t,.)\|_{H^s} \leq \|u(t,.)\|_{H^{s+1}} \sum_{i,j} \sum_{|\alpha|+|\beta|\leq s} \|D^\alpha u_i D^\beta u_j\|_{L^2} - 2\|\nabla u\|^2_{H^s} \tag{3}$$

for some positive $s \in \mathbb{R}$ (this is the standard notation for Sobolev spaces H^s which will be defined below). This type of estimate is valid also for initial-value problems on $[0,\infty)\times\Omega$ with periodic boundary conditions which are defined on the n-torus $\mathbb{T}^n$. From this point of view the incompressible Navier-Stokes equation system is a multivariate Burgers equation system (of Cauchy type) with an external force, and with the negative gradient pressure as source terms. In addition an incompressibility condition for the velocity is satisfied, i.e., the divergence of the velocity is zero. Indeed, the Navier-Stokes equation system is defined via three equations, the nonlinear diffusion equation

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \mathbf{f}_{ex} \quad t\geq 0, \quad x\in\mathbb{R}^n, \tag{4}$$

the incompressibility condition

$$\nabla\cdot\mathbf{v} = 0, \quad t\geq 0, \quad x\in\mathbb{R}^n, \tag{5}$$

and the initial condition

$$\mathbf{v}(0,x) = \mathbf{h}(x), \quad x\in\mathbb{R}^n. \tag{6}$$

The Leray projection for this equation system is obtained from a Poisson equation for the pressure, i.e., we have

$$-\Delta p = \textstyle\sum_{j,k=1}^n \left(\frac{\partial}{\partial x_k}v_j\right)\left(\frac{\partial}{\partial x_j}v_k\right), \tag{7}$$

with the solution

$$p(t,x) = -\int_{\mathbb{R}^n} K_n(x-y)\sum_{j,k=1}^n \left(\frac{\partial v_k}{\partial x_j}\frac{\partial v_j}{\partial x_k}\right)(t,y)dy. \tag{8}$$

Here,

$$K_n(x) := \begin{cases} \frac{1}{2\pi}\ln|x|, & \text{if } \ n=2, \\ \\ \frac{1}{(2-n)\omega_n}|x|^{2-n}, & \text{if } \ n\geq 3 \end{cases} \tag{9}$$

is the Poisson kernel. Here, $|.|$ denotes the Euclidean norm and ω_n denotes the area of the unit n-sphere. Since the function $x\rightarrow |x|^\mu$ is locally integrable if and only if $\mu > -n$ we observe that for $n\geq 3$ the functions

$$x\rightarrow \frac{\partial}{\partial x_l}K_n(x) = \omega_n^{-1}\frac{x_l}{|x|^n} \tag{10}$$

are integrable around $x = 0$ for $1 \leq l \leq n$ (you may observe this explicitly by writing the derivative of the kernel K_n in (10) in polar coordinates). Note that for all $x \in \mathbb{R}^n$ we have a natural upper bound

$$\left|\frac{\partial}{\partial x_l} K_n(x)\right| \leq \omega_n^{-1}, \tag{11}$$

but the derivatives of the velocity in the expression for the pressure in (8) should have some decay at spatial infinity in order that the Leray projection form makes sense. Later, we shall observe that a solution $\mathbf{v}$ can be constructed which decays at spatial infinity such that the integral on the right side of (8) exists globally. More precisely, we shall see that for initial data $\mathbf{h} = (h_1, \cdots, h_n)^T$ which are in Sobolev spaces $[H^s(\mathbb{R}^n)]^n$ for s large enough (i.e., for $h_i \in H^s(\mathbb{R}^n)$ for $1 \leq i \leq n$) we have a certain decay at spatial infinity of the solution such that

$$\int_{\mathbb{R}^n} \sum_{j,k=1}^{n} \left| \left(\frac{\partial v_k}{\partial x_j}\frac{\partial v_j}{\partial x_k}\right)(t,y)\right| dy < \infty. \tag{12}$$

Note that a global classical solution justifies that we speak of 'the' solution. Hence, it makes sense to define the Navier-Stokes Cauchy problem for divergence free velocity fields by

$$\left\{\begin{array}{l} \frac{\partial v_i}{\partial t} - \nu \sum_{j=1}^{n} \frac{\partial^2 v_i}{\partial x_j^2} + \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} = \\ \\ \qquad \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^{n} \left(\frac{\partial v_k}{\partial x_j}\frac{\partial v_j}{\partial x_k}\right)(t,y)dy, \\ \\ \mathbf{v}(0,.) = \mathbf{h}, \end{array}\right. \tag{13}$$

for $1 \leq i \leq n$, and where external forces $\mathbf{f}_{ex}$ are set to zero for simplicity. In order to have polynomial decay at spatial infinity of the data $\mathbf{h}$ we take a standard assumption that we have $\mathbf{h} \in [H^s]^n$ for all $s \in \mathbb{R}$. This assumption may be weakened a bit (this may depend on the regularity which you want to achieve- cf. below), but from an algorithmic or physical perspective it is satisfying. Note that here and in the following we write

$$\left(\frac{\partial v_k}{\partial x_j}\frac{\partial v_j}{\partial x_k}\right)(t,y) := \frac{\partial v_k}{\partial x_j}(t,y)\frac{\partial v_j}{\partial x_k}(t,y) \tag{14}$$

for the sake of brevity (similar for sums of functions etc.). We look for classical solutions in the space of divergence free vector fields, i.e., in the space

$$\left\{\mathbf{v} \in \left[C_b^{1,2}\left([0,\infty) \times \mathbb{R}^n\right)\right]^n \mid \operatorname{div} \mathbf{v} = 0\right\}, \tag{15}$$

where the solution $\mathbf{v}$ should be bounded and regular (i.e. derivatives should exists in a classical sense such that uniqueness is guaranteed), and where

some decay at spatial infinity guarantees that (12) is satisfied. Here, the function space $C^{1,2}_b\left([0,\infty)\times\mathbb{R}^n\right)$ is the space of scalar functions which have bounded time derivatives up to first order and bounded spatial derivatives up to second order, and the function space $\left[C^{1,2}_b\left([0,\infty)\times\mathbb{R}^n\right)\right]^n$ is the space of vector-valued functions $\mathbf{v} := (v_1,\cdots,v_n)^T$ with components $v_i \in C^{1,2}_b\left([0,\infty)\times\mathbb{R}^n\right)$ for all $1\leq i\leq n$. Here, and in the following we write vector-valued functions in boldface letters. From this point of view it seems natural to measure the construction of the Navier-Stokes solution $\mathbf{v} = (v_1,\cdots,v_n)$ in a $|.|_{1,2}$-norm of functions with globally bounded time derivatives up to first order and spatial derivatives up to second order (for each component function v_i) plus an integral norm related to (12). Note that the norm $|.|_{1,2}$ does not lead to a Banach space, i.e., the limit of successive approximations with finite $|.|_{1,2}$ norms will not have a finite $|.|_{1,2}$-norm.

The incompressible Navier-Stokes equation cannot be solved by a simple global fixed point iteration. Similar as in the case of the multidimensional Burgers equation (cf. [6]) we choose a time-discretized scheme and construct fixed points which are local in time. However, compared to the multidimensional Burgers equation controlling the growth of the integral term in (13) is an additional difficulty. For $l\geq 1$ let

$$\mathbf{v}^{\rho,l} = \left(v^{\rho,l}_1,\cdots,v^{\rho,l}_n\right)^T : [l-1,l]\times\mathbb{R}^n \to \mathbb{R}^n \tag{16}$$

be a solution of (13) on the domain $[l-1,l]\times\mathbb{R}^n$ in transformed time coordinates τ with

$$t = \rho_l\tau \tag{17}$$

and with initial data $\mathbf{v}^{\rho,l}(l-1,.) = \mathbf{v}^{\rho,l-1}(l-1,.)$ being the final data of the previous time step number $l-1$, where $\mathbf{v}^{\rho,1}(0,.) = \mathbf{h}(,.)$. Note that $\mathbf{v}^{\rho,l}(l-1,.) = \mathbf{v}\left(\sum_{m=1}^{l-1}\rho_m,.\right)$ for $l\geq 1$. Here, the idea is that choices $\rho_l < 1$ small enough lead to a local converging iteration scheme in transformed coordinates τ on domains $[l-1,l]\times\mathbb{R}^n$ while on the other side the numbers ρ_l may be chosen large enough such that the sum of time-step size ρ_l diverges. The latter condition implies that the scheme is global in time (the discussion here is preliminary; cf. below for a more extensive discussion of this time discretization). It seems difficult to control the growth of the solutions $v^{\rho,l}_i$ directly if the time step size series $(\rho_l)_l$ is large enough such that

$$\sum_{l=1}^{N}\rho_l \uparrow \infty \quad \text{as} \quad N\uparrow\infty. \tag{18}$$

However, the requirement (18) for a global scheme is weaker than the requirement of a scheme where the time step size is bounded from below independently of the time step number l. Indeed the requirement (18) allows

us to choose time step sizes ρ_l of order

$$\rho_l \sim \frac{1}{l} \tag{19}$$

in order to define a global scheme. This does not mean that we shall construct a solution which has a linear bound with respect to time. Indeed we shall construct a bounded solution $\mathbf{v}$ where all components v_i are gobally bounded with respect to the $|.|_{1,2}$ norm. However, choosing time step sizes of order (19) is useful in order to control the integral magnitude (12). However, linear growth of the solution with respect to a supremum norm up to a certain order of derivatives is not sufficient in order to define a global scheme.

Therefore we shall construct a bounded solution of an equivalent problem with solution $\mathbf{v}^{r,\rho}$ defined recursively via a series $\mathbf{v}^{r,\rho,l}$. Here 'bounded' means bounded with respect to the supremum norm. For the integral magnitude 12 it suffices to have linear growth in time. The choice of the time step size 19 will ensure the local convergence of the scheme. Here for each $l \geq 1$ the functions $\mathbf{v}^{r,\rho,l},\ \mathbf{v}^{\rho,l},\ \mathbf{r}^l$ are all defined on the domain $[l-1,l]\times\mathbb{R}^n$. For $\tau = l-1$ we have $\mathbf{v}^{r,\rho,l} = \mathbf{v}^{r,\rho,l-1}$ for all $l \geq 1$. The series $\left(\mathbf{r}^l\right)_l$ is designed in order to control the source terms of the equations for $\mathbf{v}^{r,\rho,l}$. For integers $l \geq 1$ (time-steps) we shall construct a series of real numbers (ρ_l) and a family of recursively defined functions $\left(r_i^l\right)$, $l \in \mathbb{N},\ 1 \leq i \leq n$, where

$$r_i^l : [l-1,l]\times\mathbb{R}^n \to \mathbb{R}, \tag{20}$$

and consider the equation system for the functions

$$v_i^{r,\rho,l} = v_i^{\rho,l} + r_i^l. \tag{21}$$

It is not sufficient to construct a global upper bound for $\mathbf{v}^\rho + \mathbf{r}$ of course. We shall construct a global upper bound for $\mathbf{r} = (\mathbf{r}^l)_{l\geq 1} = (r_1^l,\cdots,r_n^l)_{l\geq 1}$ and a global upper bound for $\mathbf{v}^{r,\rho}$ and shall conclude that $\mathbf{v} = \mathbf{v}^\rho = \mathbf{v}^{r,\rho} - \mathbf{r}$ has itself a global upper bound. For $l \geq 1$ the functions $r_i^l,\ 1 \leq i \leq n$ (which determine a global function $\mathbf{r}$) satisfy themselves for each time step number l a multivariate partial differential equation system which is determined dynamically within the solution scheme for $\mathbf{v}^{r,\rho}$ and $\mathbf{r}$. In order to explain this in more detail we first consider the equations for the functional series $\mathbf{v}^{r,\rho,l},\ l \geq 1$ for a rather arbitrary class of functions $r_i^l \in C_b^{1,2}\left([l-1,l]\times\mathbb{R}^n\right)$, i.e., for all $1 \leq i \leq n$ the functions $(\tau,x) \to r_i^l(\tau,x)$ are assumed to be bounded functions with a bounded time derivative (in a weak sense at the integer values $l \geq 1$ and in a classical sense elsewhere) and bounded spatial derivatives up to second order on the domain $(l-1,l]\times\mathbb{R}^n$. Furthermore, the functions are defined recursively with respect to the time step number l. For $\mathbf{r} = (r_1,\cdots,r_n)^T : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ we have

$$r_i(\tau,x) = r_i^l(\tau,x) \text{ iff } (\tau,x) \in [l-1,l]\times\mathbb{R}^n. \tag{22}$$

We construct global functions $\mathbf{v}^{r,\rho} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{r} : [0,\infty) \times \mathbb{R}^n$ time step by time step. First we prove that for some bounded function $\mathbf{r}$ the function $(t,x) \to \mathbf{v}^r(t,x) = \mathbf{v}^{r,\rho}(\tau,x)$ is bounded and a solution of a equation system which is equivalent to the Navier-Stokes equation system. Then we conclude that there is a solution of the Navier-Stokes equation system $\mathbf{v} = \mathbf{v}^r + \mathbf{r}$ which is bounded and in $C^{1,2}\left([0,\infty) \times \mathbb{R}^n\right)$ (the space of functions which have classical time derivatives of first order and classical spatial derivatives of second order). Bounded classical solutions with polynomial decay at infinity lead to further regularity and then to uniqueness. Indeed, it is well-known that energy estimates imply that the Navier Stokes equation 'determines' $\mathbf{v}$ among the smooth solutions. Next for each time step number l the restriction $\mathbf{v}^{r,\rho,l}$ of the function $\mathbf{v}^{r,\rho}$ to the domain $[l-1,l] \times \mathbb{R}^n$, i.e., the function

$$\begin{array}{ll} \mathbf{v}^{r,\rho,l} = \left(v_1^{r,\rho,l}, \cdots, v_n^{r,\rho,l}\right)^T : [l-1,l] \times \mathbb{R}^n \to \mathbb{R}^n \\ \\ v_i^{r,\rho,l} = v_i^{\rho,l} + r_i^l, \end{array} \tag{23}$$

satisfies

$$\left\lbrace \begin{array}{ll} \frac{\partial v_i^{r,\rho,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l} \frac{\partial v_i^{r,\rho,l}}{\partial x_j} = L_i^{\rho,l}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l}) + \\ \\ \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,l}}{\partial x_j} \frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy + r_{i,\tau}^l, \\ \\ \mathbf{v}^{r,\rho,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.), \end{array} \right. \tag{24}$$

where

$$\begin{array}{ll} L_i^{\rho,l}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l}) \equiv -\rho_l \nu \Delta r_i^l + \rho_l \sum_{j=1}^n r_j^l \frac{\partial r_i^l}{\partial x_j} \\ \\ +\rho_l \sum_{j=1}^n r_j^l \frac{\partial v_i^{r,\rho,l}}{\partial x_j} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l} \frac{\partial r_i^l}{\partial x_j} \\ \\ -2\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial r_k^l}{\partial x_j} \frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy \\ \\ -\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial r_k^l}{\partial x_j} \frac{\partial r_j^l}{\partial x_k}\right)(\tau,y)dy. \end{array} \tag{25}$$

Note that the functional $L_i^{\rho,l}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l})$ is affine with respect to s $\mathbf{v}^{r,\rho,l}$ and its first derivatives. Sometimes it is useful to consider the linear part $L_i^{\rho,l,0}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l})$

of the functional $L_i^{\rho,l}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l})$. Therefore we write

$$
\begin{array}{ll}
L_i^{\rho,l}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l}) =: -\rho_l\nu\Delta r_i^l+\rho_l\sum_{j=1}^n r_j^l\frac{\partial r_i^l}{\partial x_j}\\
\\
-\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial r^l_j}{\partial x_k}\right)(\tau,y)dy+L_i^{\rho,l,0}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l})\\
\\
=: S^{\rho,l}_i\left(\mathbf{r}^l\right)+L_i^{\rho,l,0}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l}).
\end{array}
\tag{26}
$$

Note that the operators $L^{\rho,l}_i$ and $L^{\rho,l,0}_i$ are determined by the relation (23) and the Navier-Stokes equation system. Furthermore, in (24) we have $\mathbf{v}^{r,\rho,0}(0,.)=\mathbf{h}$ in case $l=1$. Note that $L_i^{\rho,l,0}(\mathbf{r}^l;\mathbf{v}^{r,\rho,l})$ is a linear differential operator with respect to the function $\mathbf{v}^{r,\rho,l}$. Note that all terms of the functional $L^{\rho,l}_i$ have a factor ρ_l. The numbers ρ_l measure the time step size in original coordinates and will be chosen such that at each time step we can construct a local time solution of the incompressible Navier-Stokes equation by an iteration procedure where we use polynomial decay at spatial infinity of the iterative approximations of the solution. Furthermore, the functions $\mathbf{r}^l$ are chosen such that at each time step l that we can control the source terms on the right side of (24) depending on the computation of the data from the last time step. This is a crucial idea of our method. More specifically

α) the choice of ρ_l ensures the time-local convergence of an iteration scheme which computes approximations $\mathbf{v}^{r,\rho,k,l}$ for the local solution $\mathbf{v}^{r,\rho,l}$ of the incompressible Navier-Stokes equation at each time step l, i.e., the solution on the domain $[l-1,l]\times{\mathbb R}^n$ with respect to transformed time coordinates τ. For each time step number $l\geq 1$ the local iteration determines the function $\mathbf{v}^{r,\rho,l}$ as a limit of the functional series $\mathbf{v}^{r,\rho,k,l},k\geq 0$. For given $\mathbf{r}$ and each $k\geq 0$ the function $\mathbf{v}^{r,\rho,k,l}$ satisfies

$$
\left\lbrace\begin{array}{ll}
\frac{\partial v^{r,\rho,k,l}_i}{\partial \tau}-\rho_l\nu\sum_{j=1}^n \frac{\partial^2 v^{r,\rho,k,l}_i}{\partial x_j^2}
+\rho_l\sum_{j=1}^n v^{r,\rho,k-1,l}_j\frac{\partial v^{r,\rho,k,l}_i}{\partial x_j}\\
\\
=L_i^{\rho,l}(\mathbf{r}^l;\mathbf{v}^{r,\rho,k-1,l})+\\
\\
\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v^{r,\rho,k-1,l}_m}{\partial x_j}\frac{\partial v^{r,\rho,k-1,l}_j}{\partial x_m}\right)(\tau,y)dy+r^l_{i,\tau},\\
\\
\mathbf{v}^{r,\rho,k,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.).
\end{array}\right.
\tag{27}
$$

Here, in case of $k=0$ (resp. $k-1=-1$) we define

$$
\mathbf{v}^{r,\rho,-1,l}(\tau,x)=\mathbf{v}^{r,\rho,l-1}(l-1,x) \mbox{ for all } (\tau,x)\in [l-1,l]\times {\mathbb R}^n. \tag{28}
$$

We shall determine a the time step size

$$\rho_l = \frac{C}{l} \tag{29}$$

for some constant $C > 0$ where all local schemes for $\mathbf{v}^{r,\rho,k,l}$ converge while $\sum_{l=1}^{M} \rho_l \uparrow \infty$ as $M \uparrow \infty$. The latter property of the time step sizes makes the scheme global. There is a difference to the previous scheme for the multivariate Burgers equation which has a time step size of order $1/l$ (cf. [6]). In the present scheme we use the time step size (29) in order to keep the additional integral term of the Navier Stokes equation ('additional' compared to the multidimensional Burgers equation) under control. Note that it is only for analytical reasons that we use a decreasing time step size of this form in order to make the scheme global (because the sum of time step sizes goes to infinity as the number of time steps goes to infinity). A scheme with increasing time step size is preferable from a numerical/computational point of view. Once we have proved that the solution function $\mathbf{v}$ is bounded itself we can alter the scheme for numerical purposes to a scheme with increasing time step size.

β) In order to determine the function $\mathbf{r}^l$ for each l recursively we assume that $\mathbf{v}^{r,l-1}(l-1,.)$ and $\mathbf{r}^{l-1}(l-1,.)$ have been computed (in case $l=1$ the function $\mathbf{v}^{r,0}(l-1,.)$ equals the initial data function $\mathbf{h}$ which is given, and $\mathbf{r}^0$ will be set to zero). Given this information of the previous time step we first determine a function ϕ^l_i which is constructed in order to control the growth of the functions r^l_i. Furthermore, the functions ϕ^l_i and r^l_i are constructed in order to control the growth of the local Navier-Stokes solution $v_i^{r,\rho,l}$ and r^l_i at time step l. At each time step we control the growth of the local solution $\mathbf{v}^{r,\rho,l}$ as a limit of an iteration procedure of successive approximations

$$\mathbf{v}^{r,\rho,k,l} = \mathbf{v}^{r,\rho,0,l} + \sum_{m=1}^{k} \left(\mathbf{v}^{r,\rho,m,l} - \mathbf{v}^{r,\rho,m-1,l} \right), \tag{30}$$

where we control the growth of the successive approximations of this series, and then show that this control is preserved in the limit $k \uparrow \infty$. The advantage of this approach is that all members of the series in (30) are solutions of linear equations. At time step $l \geq 1$ we compute the function $\mathbf{r}^l$ in terms of the data of the previous time step first. Having determined this function $\mathbf{r}^l$ appropriately we show that the growth of the function $\mathbf{v}^{r,\rho,0,l}$ is controlled. Then we shall show that for small time step size ρ_l the sum with the summands $\delta\mathbf{v}^{r,\rho,k,l} = \left(\mathbf{v}^{r,\rho,m,l} - \mathbf{v}^{r,\rho,m-1,l}\right)$ of (30) is small enough such that some growth property established for $\mathbf{v}^{r,\rho,0,l}$ is preserved for all approximations $\mathbf{v}^{r,\rho,k,l}$ and in the limit for $\mathbf{v}^{r,\rho,l}$. At the same time all time step

size numbers ρ_l have a lower bound of form (29). Hence, the scheme is global. We exploit that we have some freedom in the choice of the functions $\mathbf{r}^l$. The construction is such that the global function $\mathbf{r}$ which equals the local function $\mathbf{r}^l$ on each domain $[l-1,l]\times\mathbb{R}^n$ is bounded. Each $\mathbf{r}^l$ is itself a solution of a linear equation with with certain source terms ϕ^l_i which serve as 'consumption terms' in certain regions of the domain $[l-1,l]\times\mathbb{R}^n$ and control the growth of the functions for $\mathbf{v}^{r,\rho,0,l}$ and $\mathbf{r}^l$. For each $1\leq i\leq n$ the function ϕ^l_i is determined in terms of the information gained at the previous time step, i.e., in terms of the functions $x\rightarrow r^{l-1}_i(l-1,x)$ and $x\rightarrow v^{r,\rho,l-1}_i(l-1,x)$. For simplicity of notation we denote these functions by $r^{l-1}_i(l-1,.)$ and $v^{r,\rho,l-1}_i(l-1,.)$ sometimes. The functions ϕ^l_i are constructed grosso modo as follows (details will be given in the proof of the main theorem below). For each $1\leq i\leq n$ and at each time step l the function ϕ^l_i is constructed as a sum $\phi^l_i=\phi^{v,l}_i+\phi^{r,l}_i$ in two steps. The function $\phi^{v,l}_i$ is constructed in terms of $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ in order to control the growth of the function $\mathbf{v}^{r,\rho}$. Similarly, the function $\phi^{r,l}_i$ is constructed in terms of $\mathbf{r}$ and the function $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ in order to control the growth of the function $\mathbf{v}^{r,\rho}$. Assume that for $1\leq i\leq n$ we have

$$|v^{r,\rho,l-1}_i(l-1,.)|_0=\sup_{x\in\mathbb{R}^n}|v^{r,\rho,l-1}_i(l-1,.)|\leq C \tag{31}$$

for some $C>0$. We consider two sets where the modulus of these data exceeds a certain level. Let

$$D^{v,l-1}_{+,i}:=\left\{x|v^{r,\rho,l-1}_i(l-1,x)\in\left[\frac{C}{2},C\right]\right\}, \tag{32}$$

and let

$$D^{v,l-1}_{-,i}:=\left\{x|v^{r,\rho,l-1}_i(l-1,x)\in\left[-C,-\frac{C}{2}\right]\right\}. \tag{33}$$

Then there are several ways to define functions $\phi^{v,l}_i$ on domains $(l-1,l]\times\mathbb{R}^n$ which will serve as source terms and 'act against the tendency of becoming large' which is encoded in the sets $D^{v,l-1}_{+,i}$ and $D^{v,l-1}_{-,i}$ to some extent. Maybe the most simple form is to define these function $\phi^{v,l}_i$ in a form with no real time dependence. (This has some slight disadvantages because we have to do some extra work concerning joint regularity of the solution functions $\mathbf{v}^{r,\rho,l}$ at the discrete times l, but we can deal with these problems without too much difficulties (cf. discussion in the following remark below). Hence, we shall define (details are given in the proof of the main theorem)

$$\begin{aligned}&\phi^{v,l}_i:(l-1,l]\times\mathbb{R}^n\rightarrow\mathbb{R}\\&\phi^{v,l}_i(\tau,x):=\phi^{v,s,l}_i(x),\end{aligned} \tag{34}$$

where $\phi_i^{v,s,l} \in C_b^{1,2}$ is determined such that

$$\phi_i^{v,s,l}(x) = \begin{cases} -1 \ \text{ if } x \in D_{i,+}^{v,l-1}, \\ \\ 1 \ \text{ if } x \in D_{i,-}^{v,l-1}, \end{cases} \tag{35}$$

and such that

$$\sup_{x \in \mathbb{R}^n} |\phi_i^{v,s,l}(x)| \leq 1. \tag{36}$$

This is only grosso modo what is going on. The uniform upper bound for the functions ϕ_i^l with respect to the $|.|_{1,2}$-norm ensures that the distance

$$\text{dist}\left(D_{i,+}^{v,l-1}, D_{i,-}^{v,l-1}\right) \geq c \tag{37}$$

for some constant $c > 0$ which is independent of the time step number l. Furthermore, we shall extend the definition of the functions ϕ_i^l to the whole domain $[l-1, l] \times \mathbb{R}^n$ such that

$$|\phi_i^l(\tau, .)|_{L^2} \leq C_n^* \left(C_r + lC_r\right) \tag{38}$$

uniformly in the time variable $\tau \in [l-1, l]$, where L^2 denotes the usual Hilbert space of squared integrable functions. Indeed, we shall have related $H^2 \equiv H^2\left(\mathbb{R}^n\right)$-estimates for the functions $\mathbf{r}^l$ and $\mathbf{v}^{r,\rho,l}$, where H^2 denotes the Hilbert space of functions with L^2 derivatives up to second orderNote that the estimate in (38) is linear with respect to the time step number l. Therefore our choice of $\rho_l \sim \frac{1}{l}$ is appropriate. The estimate (38) is closely related to the estimation of the term

$$\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i} K_n(x-y) \sum_{j,k=1}^n \left(\frac{\partial v_k}{\partial x_j} \frac{\partial v_j}{\partial x_k}\right)(\tau, y) dy \tag{39}$$

in the Leray projection of the Navier stokes equation. Indeed, estimates of the form (38) for ϕ_i^l imply that we have a similar linear upper bound for (39) in case $n = 3$. In the case of $n > 3$ our method may be adapted by use of certain Banach space norms $H^{s,p}$.

Next for the construction of $\phi_i^{r,l}$ assume that for $1 \leq i \leq n$ we have

$$\sup_{x \in \mathbb{R}^n} |r_i^{l-1}(l-1, .)| \leq C_r^0, \tag{40}$$

where C_r^0 will be a constant which depends only on the initial data dimension and viscosity. We shall construct $\mathbf{r}^l$ such that $|r_i^l|_0 \leq C_r^0$ with respect to the supremum norm $|.|_0$, and for all $l \geq 1$, and all $1 \leq i \leq n$. Let

$$\begin{array}{l} D_{+,i}^{r,l-1} := \\ \\ \left\{ x | r_i^l(l-1, x) \in \left[\frac{C_r^0}{2}, C_r^0\right] \ \right\}, \end{array} \tag{41}$$

and let

$$D^{r,l-1}_{-,i} := \left\{ x | r^l_i(l-1,x) \in \left[-C^0_r, -\frac{C^0_r}{2} \right] \right\}. \tag{42}$$

$$\begin{array}{ll} \phi^{r,l}_i : (l-1,l] \times \mathbb{R}^n \to \mathbb{R} \\ \phi^{r,l}_i(\tau,x) := \phi^{r,s,l}_i(x), \end{array} \tag{43}$$

where $\phi^{r,s,l}_i \in C^{1,2}_b$ is determined such that

$$\phi^{r,s,l}_i(x) = \begin{cases} -1 \ \text{ if } x \in D^{r,l-1}_{i,+}, \\ 1 \ \text{ if } x \in D^{r,l-1}_{i,-}, \end{cases} \tag{44}$$

and such that

$$\sup_{x \in \mathbb{R}^n} |\phi^{r,s,l}_i(x)| \leq 1. \tag{45}$$

Furthermore the extension of ϕ^l_i to the wholedomain is such that the H^1 norm has a linear bound with respect to the time step number l

Remark 1.1. The definition of the source term functions ϕ^l_i is such that we may have bounded jumps at the time discretization points l of the scheme. However, in the representation of the functions $v^{r,\rho,0,l}_i$ and r^l_i the bounded source terms become integrated over time starting from $\tau = l-1$ at each time step l. This will ensure that the construction of the global solution function $\mathbf{v}^{r,\rho}$ and $\mathbf{r}^l$ are uniformly bounded continuous over time. We shall see that this is sufficient for us. However it is also possible to construct global functions ϕ_i which equal ϕ^l_i on each domain $(l-1,l] \times \mathbb{R}^n$ which are differentiable from the beginning defining

$$\begin{array}{ll} \phi^{v,l}_i : [l-1,l] \times \mathbb{R}^n \to \mathbb{R} \\ \phi^{v,l}_i(\tau,x) := \sin^2\left(\pi(\tau-(l-1))\right) \phi^{v,s,l}_i(x), \end{array} \tag{46}$$

and

$$\begin{array}{ll} \phi^{r,l}_i : [l-1,l] \times \mathbb{R}^n \to \mathbb{R} \\ \phi^{r,l}_i(\tau,x) := \sin^2\left(\pi(\tau-(l-1))\right) \phi^{r,s,l}_i(x), \end{array} \tag{47}$$

Then we get global bounded solution functions $\mathbf{v}^{r,\rho}$ and $\mathbf{r}$ which are differentiable across time points $\tau = l$. However the growth control of the local functions $\mathbf{v}^{r,\rho,l}$ and $\mathbf{r}^l$ becomes a little bit cumbersome. Hence we proceed with the definition which leads to global solutions which are bounded continuous at time points $\tau = l$ first.

We continue to outline the main ideas within item β). The functions ϕ_i^l serve as source terms or 'consumption terms' in the following parabolic equation for $\mathbf{r}^l$ (which is a linearising of a time-local Navier-Stokes type equation with source term ϕ_i^l among other source terms), where for $1 \leq i \leq n$ we have

$$\begin{cases} r_{i,\tau}^l - \rho_l \nu \Delta r_i^l + \rho_l \sum_{j=1}^n r_j^{l-1}(l-1,.)\frac{\partial r_i^l}{\partial x_j} = \\ \\ +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial r_k^{l-1}}{\partial x_j}\frac{\partial r_j^{l-1}}{\partial x_k}\right)(l-1,y)dy \\ \\ -L_i^{\rho,l,0}(\mathbf{r}^{l-1}(l-1,.);\mathbf{v}^{r,\rho,l-1}(l-1,.)) \\ \\ -\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,m=1}^n \left(\frac{\partial v_m^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_m}\right)(l-1,y)dy + \phi_i^l \\ \\ r_i^l(l-1,.) = r_i^{l-1}(l-1,.), \end{cases} \tag{48}$$

and where

$$L_i^{\rho,l,0}(\mathbf{r}^{l-1};\mathbf{v}^{r,\rho,l-1}) \equiv +\rho_l \sum_{j=1}^n r_j^{\rho,l-1}\frac{\partial v_i^{r,\rho,l-1}}{\partial x_j} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l-1}\frac{\partial r_i^{l-1}}{\partial x_j}$$

$$-2\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial r_k^{l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,y)dy. \tag{49}$$

Note the signs on the right side of (48). Note that dependence on the function $\mathbf{v}^{r,\rho}$ involves only the function $x \rightarrow \mathbf{v}^{r,\rho,l-1}(l-1,.)$ which is known at time step l. Assume that this equation has a classical solution $\mathbf{r}^l = (r_1^l, \cdots, r_n^l)$ in $C^{1,2}$ on the domain $[l-1,l] \times \mathbb{R}^n$ (we shall show existence of such classical solutions for time-local Navier-Stokes type systems below). Plugging (48) in (27) for $k=0$ (note that for $k=0$ $v_j^{r,\rho,k-1,l} = v_j^{r,\rho,-1,l} = v_j^{r,\rho,l-1}$) we get the equation

$$\begin{cases} \frac{\partial v_i^{r,\rho,0,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l-1}(l-1,.)\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j} = \phi_i^l + \delta_i^l, \\ \\ \mathbf{v}^{r,\rho,0,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.), \end{cases} \tag{50}$$

where we shall point out that δ_i^l is small compared to ϕ_i^l in the relevant regions $D_i^{v,l,-}$, $D_i^{r,l,-}$, $D_i^{v,l,+}$, and $D_i^{r,l,+}$ (provided that ρ_l is small enough).

The choice of ϕ_l is such that in these relevant regions the behavior of the solutions $\mathbf{v}^{r,\rho,0,l}$ and $\mathbf{r}^l$ is controlled. Moreover, we know that the

function

$$\mathbf{v}^{r,\rho,l} = \mathbf{v}^{r,\rho,0,l} + \sum_{k\geq 1} \delta\mathbf{v}^{r,\rho,k,l}, \tag{51}$$

is small where the sum of functions

$$\delta\mathbf{v}^{r,\rho,k,l} = \mathbf{v}^{r,\rho,k,l} - \mathbf{v}^{r,\rho,k-1,l}, \tag{52}$$

is a 'perturbation' of the function $\mathbf{v}^{r,\rho,0,l}$ if ρ_l is small We may choose ρ_l in such a way that a certain control of the behavior established for $\mathbf{v}^{r,\rho,0,l}$ is also valid for all functions $\mathbf{v}^{r,\rho,k,l},\ k \geq 1$ and for the limit $\mathbf{v}^{r,\rho,l}$. Proceeding with time step numbers l in such a way while the sum $\sum_{l=1}^{m} \rho_l$ of the time step numbers ρ_l goes to infinity ensures boundedness of the function $\mathbf{v}^{r,\rho}$. However, boundedness of the latter function does not guarantee boundedness of the difference $\mathbf{v} - \mathbf{r} = \mathbf{v}^{\rho} - \mathbf{r}$. We have to ensure that $\mathbf{r}$ is bounded, i.e., there is a global bound of all functions $\mathbf{r}^l$ independently of the time step number l. We shall see later in detail that the construction outlined can be implemented such that the functions $\mathbf{v}^{r,\rho,l}$ and $\mathbf{r}^l$ are uniformly bounded, i.e., there is a number $C > 0$ such that $|v_i^{r,\rho,l}| \leq C$ and $|r_i^l| \leq C$ for all $l \geq 1$ and $1 \leq i \leq n$. The constant $C > 0$ can be computed a priori and depends only on the dimension n, the viscosity $\nu > 0$, and the initial data $\mathbf{h}$. Moreover, the sequence $\mathbf{r}^l$ defines a globally bounded function $\mathbf{r} : [0, \infty) \times \mathbb{R}^n$ which is (weakly) differentiable with respect to time and has bounded spatial derivatives in a classical sense. The functions $\mathbf{v}^{r,\rho,l} : [l-1, l] \times \mathbb{R}^n \to \mathbb{R}^n,\ l \geq 1$ which solve the equations (24) for each l with $\mathbf{v}^{\rho,r,l}(l-1, .) = \mathbf{v}^{\rho,r,l-1}(l-1, .)$ for $l \geq 2$ and $\mathbf{v}^{\rho,r,l}(l-1, .) = \mathbf{h}(l-1, .)$ for $l = 1$ define a global function

$$\mathbf{v}^{r,\rho} := \mathbf{v}^{\rho} + \mathbf{r} : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n, \tag{53}$$

which equals $\mathbf{v}^{r,\rho,l}$ on each subdomain $[l-1, l] \times \mathbb{R}^n,\ l \geq 1$. Since $\mathbf{r}$ is bounded this function $\mathbf{v}^{\rho,r}$ satisfies a system of equations equivalent to the Navier-Stokes equation system and a bounded global solution of the Navier-Stokes equation system is given by

$$\mathbf{v}(t, .) := \mathbf{v}^{r,\rho}(\tau, .) - \mathbf{r}(\tau, .). \tag{54}$$

The regularity $\mathbf{v} \in \left[C^{1,2}\left([0, \infty) \times \mathbb{R}^n\right)\right]^n$ is easily obtained if $\mathbf{v}^{r,\rho}$ and $\mathbf{r}$ are globally Hölder continuous and have a certain decay at infinity. This leads to uniqueness.

However, before we get involved with the construction of the sequence (ρ_l) and $(\mathbf{r}^l)$ in detail let us consider (13) again. From the point of view that the incompressible Navier-Stokes equation is an extension of the multivariate Burgers equation a solution of the equation (13) is an element of the

class of divergence-free solutions of a certain family of multivariate Burgers equations where for each function $\mathbf{f}$ we add a certain source term to (1) involving functions $\mathbf{f} \in \left[C^{1,2}_b\left([0,\infty)\times\mathbb{R}^n\right)\right]^n$. More precisely, consider the family $\left(\mathbf{v}^f\right)_{\mathbf{f}}$ where for each fixed $\mathbf{f}$ the function $\mathbf{v}^f$ satisfies the Cauchy problem

$$\begin{cases} \frac{\partial v^f_i}{\partial t}-\nu\sum_{j=1}^n \frac{\partial^2 v^f_i}{\partial x_j^2}+\sum_{j=1}^n v^f_j\frac{\partial v^f_i}{\partial x_j}= \\ \qquad\qquad \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial f_k}{\partial x_j}\frac{\partial f_j}{\partial x_k}\right)(t,y)dy, \\ \mathbf{v}^f(0,.)=\mathbf{h}. \end{cases} \tag{55}$$

Then a fixed point of the map (defined on regular divergence free vector fields with values in divergence free vector fields depending on time)

$$F_{\mathrm{loc}}:\mathbf{f}\to\mathbf{v}^f,\ \text{where}\ \operatorname{div}\mathbf{f}=0,\ \text{and}\ \operatorname{div}\mathbf{v}^f=0 \tag{56}$$

is a solution of the incompressible Navier-Stokes system. Let us denote a fixed point of the latter map by

$$\mathbf{v}^*=\mathbf{v}^{v^*}. \tag{57}$$

Formally, such a fixed point has a representation in terms of the fundamental solution Γ^* of the scalar equation

$$\frac{\partial\Gamma^*}{\partial t}-\nu\sum_{j=1}^n\frac{\partial^2\Gamma^*}{\partial x_j^2}+\sum_{j=1}^n v^*_j\frac{\partial\Gamma^*}{\partial x_j}=0, \tag{58}$$

where we denote $\mathbf{v}^*=(v^*_1,\cdots,v^*_n)^T$, i.e., we have the formal representation $(1\le i\le n)$

$$\begin{array}{l} v^*_i(t,x)=\int_{\mathbb{R}^n}h_i(y)\Gamma^*(t,x,0,y)dy+ \\ \int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(y-z)\right)\sum_{j,k=1}^n\left(\frac{\partial v^*_k}{\partial x_j}\frac{\partial v^*_j}{\partial x_k}\right)(s,z)\Gamma^*(t,x,s,y)dzdyds. \end{array} \tag{59}$$

Note that for all $1\le i\le n$ we have the same fundamental solution Γ^* of a scalar equation involving first order coefficients $v^*_j, 1\le j\le n$ which are the same for all $1\le i\le n$. Well, if we construct such fixed points in a time-discretized scheme, then we may use the a priori estimates of the multivariate Burgers equation of type (2) and a representation of type (59) at each time step in order analyze the growth of the solution. Note that we cannot control the integral terms involving the Poisson kernel in the Leray projection form this way. Nevertheless the representation (59) gives us a first hint how the integral term can be controlled in a time-discretized scheme

using our considerations above. There is a mutual dependence of the choice of the functions r_i^l and the choice of the numbers ρ_l which ensure the local convergence in time. The choice of the latter numbers depends on the local scheme. It is not necessary to start each time step with the solution of the corresponding multidimensional Burgers equation. Indeed, an alternative way of constructing a divergence-free fixed point

$$F_{\rm glob}:\mathbf{f}\rightarrow \mathbf{v}^f \tag{60}$$

via a time-discretized scheme may be the following: locally in time the family $\left(\mathbf{v}^f\right)_{\mathbf{f}}$ of vector-valued functions $\mathbf{v}^f=\left(v_1^f,\cdots,v_n^f\right)^T$ may satisfy for each $\mathbf{f}=(f_1,\cdots,f_n)$ the equation

$$\left\lbrace\begin{array}{ll}
\frac{\partial v_i^f}{\partial t}-\nu\sum_{j=1}^n \frac{\partial^2 v_i^f}{\partial x_j^2}+\sum_{j=1}^n f_j\frac{\partial v_i^f}{\partial x_j}=\\
\\
\qquad\qquad\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial f_k}{\partial x_j}\frac{\partial v_j^f}{\partial x_k}\right)(t,y)dy,\\
\\
\mathbf{v}^f(0,.)=\mathbf{h}.
\end{array}\right. \tag{61}$$

Here, we search for a fixed point solution $\mathbf{f}^*$ with $F_{\rm glob}(\mathbf{f}^*)=\mathbf{v}^{f^*}$, where

$$\operatorname{div}\mathbf{f}^*=\operatorname{div}\mathbf{v}^{f^*}=0. \tag{62}$$

Let us consider this map more closely. Start with the equation

$$\frac{\partial \mathbf{v}^f}{\partial t}-\nu\Delta \mathbf{v}^f+(\mathbf{f}\cdot\nabla)\mathbf{v}^f=-\nabla p^f\qquad t\geq 0,\quad x\in {\mathbb R}^n, \tag{63}$$

for some scalar function p^f, and where

$$\mathbf{f}\in\left\lbrace \mathbf{u}\in\left[C^{1,2}_b\left([0,\infty)\times {\mathbb R}^n\right)\right]^n\,|\operatorname{div}\mathbf{u}=0,\ \mathbf{u}(0,.)=\mathbf{h}\right\rbrace. \tag{64}$$

In the following we denote spatial derivatives in the form $f_{i,j}:=\frac{\partial f_i}{\partial x_j}$ and $f_{i,j,k}:=\frac{\partial f_i^2}{\partial x_j\partial x_k}$ etc. as is usual for example in the literature on the theory of general relativity. Sometimes we feel free to denote time derivatives in the form $f_{i,t}=\frac{\partial f_i}{\partial t}$. For the divergence $\operatorname{div}\mathbf{v}^f$ we have

$$\frac{\partial}{\partial t}\operatorname{div}\mathbf{v}^f+\nu\Delta\operatorname{div}\mathbf{v}^f+\sum_j f_j\frac{\partial}{\partial x_j}\operatorname{div}\mathbf{v}^f=-\sum_{i,j=1}^n f_{i,j}v^f_{j,i}-\Delta p^f, \tag{65}$$

where $\operatorname{div}\mathbf{v}^f(0,.)=\operatorname{div}\mathbf{h}=0$. Now let Γ^f be the fundamental solution of

$$\frac{\partial}{\partial t}\Gamma^f-\nu\Delta\Gamma^f+\sum_{j=1}^n f_j\Gamma^f=0. \tag{66}$$

Then the solution to equation (65) with zero initial data has the representation

$$\operatorname{div}\mathbf{v}^f(t,x)=\int_0^t\int_{\mathbb{R}^n}\left(-\sum_{i,j=1}^n f_{i,j}v^f_{j,i}-\Delta p^f\right)(s,y)\Gamma^f(t,x,s,y)dyds. \quad (67)$$

Now let $\mathbf{v}^f$ be a solution of (61). Then we have the representation

$$\begin{array}{ll}v^f_i(t,x)=\int_{\mathbb{R}^n}h_i(y)\Gamma^f(t,x,0,y)dy\\ +\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-z)\right)\sum_{j,k=1}^n\left(\frac{\partial f_k}{\partial x_j}\frac{\partial v^f_j}{\partial x_k}\right)(s,z)\Gamma^f(t,x,s,y)dydzds\end{array} \quad (68)$$

Hence if we solve (61) for $\mathbf{v}^f$ in the form (68), then this is the same is solving (63) in the form

$$\begin{array}{ll}v^f_i(t,x)=\int_{\mathbb{R}^n}h_i(y)\Gamma^f(t,x,0,y)dy\\ +\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}p\right)(s,y)\Gamma^f(t,x,s,y)dyds\end{array} \quad (69)$$

along with

$$p^f(t,x)=-\int_{\mathbb{R}^n}\left(K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial f_k}{\partial x_j}\frac{\partial v^f_j}{\partial x_k}\right)(t,y)dy, \quad (70)$$

and such that

$$\operatorname{div}\mathbf{v}^f=0 \quad (71)$$

is ensured. Hence, if we find a fixed point $\mathbf{v}^*$ of the map $\mathbf{f}\rightarrow\mathbf{v}^f$, then this function satisfies a) the equation (61), and b) the equation (4) (with $\mathbf{f}_{ex}\equiv 0$) along with

$$p(t,x)=-\int_{\mathbb{R}^n}\left(K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v^*_k}{\partial x_j}\frac{\partial v^*_j}{\partial x_k}\right)(t,y)dy, \quad (72)$$

and $\operatorname{div}\mathbf{v}^*=0$, and $\mathbf{v}^*(0,.)=\mathbf{h}(.)$. The latter ansatz has the advantage that we can easily preserve the incompressibility condition at each step of approximation of the local solution, i.e., the solution at each time step. Note that we speak of 'the' local solution here in the sense of our construction, i.e., the approximating equations involved in each time step have a unique solution. Moreover, we shall prove that the local solution is a classical solution, and then it can be shown that the classical solution is indeed unique. In any case (whether we start with the multivariate Burgers equation or with a linearized Navier-Stokes equation), it seems that such a fixed point cannot be obtained by a global iteration scheme.

Recall that it is indeed sufficient to solve (13) (this is well-known). A regular solution $\mathbf{v}$ of equation (13) satisfies

$$\begin{cases} \frac{\partial v_i}{\partial t} - \nu \sum_{j=1}^n \frac{\partial^2 v_i}{\partial x_j^2} + \sum_{j=1}^n v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial}{\partial x_i} p \\ \mathbf{v}(0,.) = \mathbf{h}, \end{cases} \tag{73}$$

along with p of form (72). The associated equation for the divergence $\operatorname{div} \mathbf{v}$ has zero initial data and a source term

$$-\sum_{i,j=1}^n \frac{\partial v_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} - \Delta_x p, \tag{74}$$

which becomes zero, since

$$\sum_{i,j=1}^n \left(\frac{\partial v_j}{\partial x_i} \frac{\partial v_i}{\partial x_j} \right)(t,x) = \Delta_x \int_{\mathbb{R}^n} K_n(x-y) \sum_{j,k=1}^n \left(\frac{\partial v_k}{\partial x_j} \frac{\partial v_j}{\partial x_k} \right)(t,y) dy \tag{75}$$

for regular $\mathbf{v}$ with the indicated decay at infinity. Here, Δ_x denotes the Laplacian (where derivatives are with respect to the variables x_i), and where some regularity of the function $\mathbf{v}$ is sufficient. Note that higher order differentiability is easily obtained if a solution in $C^{1,2}_b$ has been obtained (cf. Section 4).

Back to the time discretization, the time step sizes are given by a series of positive real numbers $(\rho_l)_{l\in\mathbb{N}}$ where $\mathbb{N}$ denotes the natural numbers (starting with $l = 1$) and the size will be chosen in such a way that the iteration at each time step l converges and such that we have global convergence, i.e., the time step sizes are large enough. In order to ensure global convergence it is sufficient to have a time step size of order $\rho_l = \frac{C}{l}$ for a constant $C > 0$ which depends only on the viscosity ν, the dimension n and the initial data $\mathbf{h}$. Prima facie it seems that our recursive construction of a bounded function $\mathbf{r}$ which equals the functions r^l_i locally on $[l-1,l] \times \mathbb{R}^n$ leads us to the conclusion that a uniform lower bound for the time step size ρ_l may be possible (note that the numbers ρ_l are time step sizes from the point of view of the original time coordinates). However, we need to have a bound for the integral magnitude in (12) too, and this leads us to our choice of a decreasing time step size. From a numerical point of view we hope that the opposite may be possible, i.e., that smoothing effects may allow to increase the time step size ρ_l as time goes by, i.e., as the time step number l increases. Once we have proved that there is a bounded classical solution we may discard the control function $\mathbf{r}$ and consider this possibility. We shall ensure that the global solution $\mathbf{v}^{r,\rho}$ (which equals $\mathbf{v}^{r,\rho,l}$ on each domain $[l-1,l] \times \mathbb{R}^n$) is bounded. Since $\mathbf{r}$ is bounded, this implies that we have a bounded global solution $\mathbf{v}^\rho = \mathbf{v}^{r,\rho} + \mathbf{r}$ (here, $\mathbf{v}^\rho$ equals $\mathbf{v}^{\rho,l}$ on each domain $[l-1,l] \times \mathbb{R}^n$). Note that $\mathbf{v}(t,.) = \mathbf{v}^\rho(\tau,.)$ where the former function is a solution of the Navier Stokes

equation. Note that we construct the function $\mathbf{r}$ time step by time step together with the function $\mathbf{v}^{r,\rho}$. Furthermore, note that $\mathbf{v}^r = \mathbf{v}^{r,\rho}$, where the first function refers to the equivalent system in original time coordinate t. Accordingly $\mathbf{v}^r$ satisfies the equation

$$\left\lbrace\begin{array}{ll}
\frac{\partial v^r_i}{\partial t}-\nu\sum_{j=1}^n \frac{\partial^2 v^r_i}{\partial x_j^2}+\sum_{j=1}^n v^r_j\frac{\partial v^r_i}{\partial x_j}=L_i(\mathbf{r};\mathbf{v}^r)+\\
\\
\qquad \rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v^r_k}{\partial x_j}\frac{\partial v^r_j}{\partial x_k}\right) (t,y)dy+r_{i,t},\\
\\
\mathbf{v}^r(0,.)=\mathbf{h}(0,.),
\end{array}\right. \tag{76}$$

where

$$\begin{array}{ll}
L_i(\mathbf{r};\mathbf{v}^r)\equiv -\nu\Delta r^l_i+\sum_{j=1}^n r_j\frac{\partial r_i}{\partial x_j}\\
\\
+\sum_{j=1}^n r_j\frac{\partial v^r_i}{\partial x_j}+\sum_{j=1}^n v^r_j\frac{\partial r_i}{\partial x_j}\\
\\
-2\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k}{\partial x_j}\frac{\partial v^r_j}{\partial x_k}\right) (t,y)dy\\
\\
-\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k}{\partial x_j}\frac{\partial r_j}{\partial x_k}\right) (t,y)dy\\
\\
=: \nu\Delta r_i+\sum_{j=1}^n r_j\frac{\partial r_i}{\partial x_j}\\
\\
\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k}{\partial x_j}\frac{\partial r_j}{\partial x_k}\right) (t,y)dy+L^0_i(\mathbf{r};\mathbf{v}^r).
\end{array} \tag{77}$$

Our solution scheme computes $\mathbf{v}^r$ and $\mathbf{r}$ (in original coordinates) simultaneously, and a global solution of the

1. Navier

-Stokes equation system is then obtained by addition $\mathbf{v}=\mathbf{v}^r+\mathbf{r}$ (we use the same symbol $\mathbf{r}$ for simplicity). Note also that $\mathbf{r}(t,.)=\mathbf{r}(\tau,.)$, hence $\mathbf{v}$ is bounded since $\mathbf{v}^r$ is bounded and $\mathbf{r}$ is bounded. Note that on the domain $\left[\sum_{m=1}^{l-1}\rho_m,\sum_{m=1}^{l}\rho_m\right]\times {\mathbb R}^n$ the solution of the Navier-Stokes equation $\mathbf{v}$ equals $\mathbf{v}^{\rho,l}$ on the domain $[l-1,l]\times {\mathbb R}^n$. As already remarked dependence of the numbers ρ_l on the time step number l is for numerical purposes because the scheme with flexible time step may take advantage of the smoothing effect of the scalar densities involved in our scheme (implying larger time step sizes as time goes by (cf. remarks in section 5)).

Remark 1.2. Note that in [6] we used a scheme with time step sizes

$$\rho_l=\frac{1}{C^*_nl}, \tag{78}$$

in order to solve the multidimensional Burgers equation, where $C_n^*>0$ is a constant which does not depend on the time step number l (it depends only on the data $\mathbf{h}$, and the dimension n , and the viscosity constant $\nu>0$ and will be determined below). The scheme considered there is not sufficient in order to prove global existence of solutions for the incompressible Navier-Stokes equation since we have an additional source term which is quadratic with respect to the gradient of the velocity. The introduction of the functions r^l_i for $l\geq 1$ and $1\leq i\leq n$ is the crucial difference which makes it possible to control the source terms in the Leray projection form of the incompressible Navier-Stokes equation.

Let us have a closer look at the time transformations $t\rightarrow \tau$, and the introduction of the function $\mathbf{r}$. We consider the Cauchy problem (13) on the domain $D=[0,\infty)\times {\mathbb R}^n$ where our interest is in the case $n\geq 3$. We shall solve the Cauchy problem in subsequent time steps $l\geq 1$ on the domains

$$D_l=[T_{l-1},T_l]\times {\mathbb R}^n, \tag{79}$$

where $T_0=0$ and $T_l=T_{l-1}+\rho_l$ for $l\geq 1$. Instead of considering a scheme with small time step size in original coordinates we may consider a equidistant scheme in transformed coordinates with time step size 1, i.e.,

$$\tau\rightarrow t_l(\tau)=\rho_l\tau \quad\text{if}\quad \tau\in [l-1,l]. \tag{80}$$

The transformed domains are denoted by $D^{\tau}_l=[l-1,l]\times {\mathbb R}^n$. The original time coordinate t has an index here in (80) in order to indicate that we are actually considering infinitely many different time transformations (one for each time step number l). However, in order to keep notation simple we drop such indices in the following. The index l makes clear which domain we consider. On each domain D^{τ}_l we have a transformed Cauchy problem

$$\left\lbrace \begin{array}{ll}
\frac{\partial v^{\rho,l}_i}{\partial \tau}-\rho_l\nu\sum_{j=1}^n \frac{\partial^2 v^{\rho,l}_i}{\partial x_j^2}
+\rho_l\sum_{j=1}^n v^{\rho,l}_j\frac{\partial v^{\rho,l}_i}{\partial x_j}=\\
\\
\hspace{2cm}\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right) \sum_{j,k=1}^n\left(\frac{\partial v^{\rho,l}_k}{\partial x_j}\frac{\partial v^{\rho,l}_j}{\partial x_k}\right) (\tau,y)dy,\\
\\
\mathbf{v}^{\rho,l}(l-1,.)=\mathbf{v}^{\rho,l-1}(l-1,.).
\end{array}\right. \tag{81}$$

Hence the initial data of the Cauchy problem for $\mathbf{v}^{\rho,l}$ on the domain D^{τ}_l (the lth time step) are the final data of the Cauchy problem for $\mathbf{v}^{\rho,l-1}$ on the domain D^{τ}_{l-1}. Note that $\mathbf{v}^{\rho,1}(0,.)=\mathbf{h}$. However, as indicated above we introduce another sequence of real functions $(\mathbf{r}^l)$ and consider for each time step $l\geq 1$ and on the domain D^{τ}_l the function

$$(\tau,x)\rightarrow \mathbf{v}^{r,\rho,l}(\tau,x):=\mathbf{v}^{\rho,l}(\tau,x)+\mathbf{r}_l. \tag{82}$$

If for time step number l the function $\mathbf{v}^{\rho,l}$ solves (81), then the function $\mathbf{v}^{r,\rho,l}$ solves the equation

$$\left\lbrace \begin{array}{ll}
\frac{\partial v^{r,\rho,l}_i}{\partial \tau}-\rho_l\nu\sum_{j=1}^n \frac{\partial^2 v^{r,\rho,l}_i}{\partial x_j^2} +\rho_l\sum_{j=1}^n v^{r,\rho,l}_j\frac{\partial v^{r,\rho,l}_i}{\partial x_j}=L^{\rho,l}_i(\mathbf{r}^l,\mathbf{v}^{r,\rho,l})+\\
\\
\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right) \sum_{j,k=1}^n\left(\frac{\partial v^{r,\rho,l}_k}{\partial x_j}\frac{\partial v^{r,\rho,l}_j}{\partial x_k}\right) (\tau,y)dy+r^l_{i,\tau},\\
\\
\mathbf{v}^{r,\rho,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.).
\end{array}\right. \tag{83}$$

The sequence $(\mathbf{r}^l)$ will be chosen (recursively with respect to l) in such a way that the global growth is controlled, i.e., that the components of $\mathbf{v}^{r,\rho,l}$ are bounded independent of l. Furthermore, since $\mathbf{r}$ is bounded this implies a bound on $\mathbf{v}^{\rho,l}$. The sequence (ρ_l) is chosen in such a way that for each time step number l the local scheme for $\mathbf{v}^{r,\rho,l}$ converges. At each time step l we approximate $\mathbf{v}^{r,\rho,l}$ iteratively by functions $\mathbf{v}^{r,\rho,k,l}$ for $k\geq 0$. Indeed, at each time step l we shall construct a solution $\mathbf{v}^{r,\rho,l}$ of (83) in form of a functional series

$$\mathbf{v}^{r,\rho,l}=\mathbf{v}^{r,\rho,0,l}+\sum_{k\geq 1}\delta \mathbf{v}^{r,\rho,k,l}, \tag{84}$$

where for each $k\geq 1$ we shall have a contraction of the successive approximations

$$\mathbf{v}^{r,\rho,k,l}=\mathbf{v}^{r,\rho,0,l}+\sum_{m=1}^k\delta \mathbf{v}^{r,\rho,m,l}, \tag{85}$$

where

$$|\delta \mathbf{v}^{r,\rho,k,l}|=|\mathbf{v}^{r,\rho,k,l}-\mathbf{v}^{r,\rho,k-1,l}|\leq c|\mathbf{v}^{r,\rho,k-1,l}-\mathbf{v}^{r,\rho,k-2,l}|. \tag{86}$$

Here the symbol $|.|$ represents some appropriate norm and we have a contraction constant $c\in (0,1)$ which turns out to be independent of the time step number l. In general we shall prove a contraction property of type (86) for specific series $(\mathbf{v}^{\rho,k,l})_k$ defined as in (iii) below with a specific first element $\mathbf{v}^{\rho,0,l}$ for each time step $l\geq 1$ (alternatively, we could use the methods in (i) or (ii)) below to set up a local scheme). We have (at least) three possibilities for the local iteration (given $(\mathbf{r}^l)$ which is computed first at each step).

(i) We start for each l the iteration with the corresponding multivariate

Burgers equation, i.e. for each l the function $\mathbf{v}^{r,\rho,0,l}$ solves the equation

$$\begin{cases} \frac{\partial v_i^{r,\rho,0,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,0,l} \frac{\partial v_i^{r,\rho,0,l}}{\partial x_j} \\ \\ = L_i^{\rho,l}\left(\mathbf{r}^l(l-1,.), \mathbf{v}^{r,\rho,l-1}(l-1,.)\right) + r_{i,\tau}^l(l-1,.) + \\ \\ \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j} \frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(l-1,y)dy, \\ \\ \mathbf{v}^{r,\rho,0,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.). \end{cases} \tag{87}$$

We shall see below in which sense we have time derivatives at the points where $\tau = l-1$ for some $l \geq 1$. For the latter equation we have the a priori estimates (2), and hence a global regular solution which is bounded by the respective initial data. The initial data $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ are the final data of the previous time step which are the result of an iteration which we have to define next. We define a series of multivariate Burgers equations for $k \geq 1$ where $\mathbf{v}^{r,\rho,k,l}$ solves

$$\begin{cases} \frac{\partial v_i^{r,\rho,k,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,k,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,k,l} \frac{\partial v_i^{r,\rho,k,l}}{\partial x_j} \\ \\ = L_i^{\rho,l}\left(\mathbf{r}^l, \mathbf{v}^{r,\rho,k-1,l}\right) + r_{i,\tau}^l \\ \\ +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,k-1,l}}{\partial x_j} \frac{\partial v_j^{r,\rho,k-1,l}}{\partial x_k}\right)(\tau,y)dy, \\ \\ \mathbf{v}^{r,\rho,k,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.). \end{cases} \tag{88}$$

Then we may prove - with an appropriate choice of ρ_l, r_l-that locally this scheme leads to a fixed point.

(ii) We may linearize for each time step the equation (83) and define a sequence of function $\mathbf{v}^{r,\rho,k,l}$ which are solutions of integro-partial-differential equations, i.e., we start with the solution $\mathbf{v}^{r,\rho,0,l}$ of

$$\begin{cases} \frac{\partial v_i^{r,\rho,0,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l-1} \frac{\partial v_i^{r,\rho,0,l}}{\partial x_j} \\ \\ = L_i^{\rho,l}\left(\mathbf{r}^l, \mathbf{v}^{r,\rho,0,l}\right) \\ \\ +r_{i,\tau}^l + \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j} \frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,y)dy, \\ \\ \mathbf{v}^{r,\rho,0,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.). \end{cases} \tag{89}$$

Then for $k \geq 1$ we define recursively functions $\mathbf{v}^{\rho,k,l}$ to be solutions of

$$\begin{cases} \frac{\partial v_i^{r,\rho,k,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,k,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,k-1,l} \frac{\partial v_i^{r,\rho,k,l}}{\partial x_j} \\ = L_i^{\rho,l}\left(\mathbf{r}^l, \mathbf{v}^{r,\rho,k-1,l}\right) + r_{i,\tau}^l \\ +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,k-1,l}}{\partial x_j} \frac{\partial v_j^{r,\rho,k,l}}{\partial x_k}\right)(\tau,y)dy, \\ \mathbf{v}^{r,\rho,k,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.). \end{cases} \tag{90}$$

Note that this is an iteration of global equations, i.e., linear partial integro-differential equations.

(iii) Alternatively, we may define a sequence of functions $\mathbf{v}^{r,\rho,k,l}$ which solve linear parabolic partial differential equations. We may choose an iteration starting with $\mathbf{v}^{r,\rho,0,l}$ which solves

$$\begin{cases} \frac{\partial v_i^{r,\rho,0,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l-1} \frac{\partial v_i^{r,\rho,0,l}}{\partial x_j} \\ = L_i^{\rho,l}\left(\mathbf{r}^l, \mathbf{v}^{r,\rho,l-1}\right) + r_{i,\tau}^l \\ +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j} \frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,y)dy \\ \mathbf{v}^{r,\rho,0,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.). \end{cases} \tag{91}$$

Then for $k \geq 1$ we define recursively functions $\mathbf{v}^{r,\rho,k,l}$ to be solutions of

$$\begin{cases} \frac{\partial v_i^{r,\rho,k,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,k,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,k-1,l} \frac{\partial v_i^{r,\rho,k-1,l}}{\partial x_j} \\ = L_i^{\rho,l}\left(\mathbf{r}^l, \mathbf{v}^{r,\rho,k-1,l}\right) \\ +r_{i,\tau}^l + \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,k-1,l}}{\partial x_j} \frac{\partial v_j^{r,\rho,k-1,l}}{\partial x_k}\right)(\tau,y)dy, \\ \mathbf{v}^{r,\rho,k,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.). \end{cases} \tag{92}$$

Let us discuss the different advantages/disadvantages of the three approaches for the local solution at each time step $l \geq 1$.

ad(i) The solution of the multivariate Burgers equation is closer to the solution of the Navier-Stokes equation than the linear approximations in

(ii) and (iii). However, at each substep k of the lth time step the function $\mathbf{v}^{\rho,k,l}$ may not be divergence-free (a divergence free vector field is obtained in the limit). We have a priori estimates (essentially a maximum principle) for the multivariate Burgers equation. On the other hand from a numerical point of view linearized equation are preferable in iteration schemes.

ad(ii) This iteration has the advantage that we can ensure that at each time step $l \geq 1$ each approximation we can ensure that $\mathbf{v}^{\rho,k,l}$ is divergence free, i.e, $\operatorname{div} \mathbf{v}^{\rho,k,l} = 0$ for all $k \geq 0$. However, at each substep k of a time step l we have to solve linear partial integro-differential equations which may be complicated from a numerical point of view.

ad(iii) In this case we have to solve local scalar linear parabolic equations in order to determine $\mathbf{v}^{\rho,k,l}$ at each substep k of a time step l. From a numerical point of view this is interesting since we have good approximations of solutions for the involved parabolic equations where the second order terms form a Laplacian (cf. ([10]), [4]). On the other hand, the local iteration does not take place in a space of divergence-free vector fields in general, i.e. a divergence free vector field is obtained in the limit.

In this paper we choose the third alternative approach for the local solutions and construct for each time step l a fixed point in some appropriate function space of maps on domains D_l^τ. This allows us to define the most efficient algorithm among the three alternatives in section 4. Compared to standard discretization schemes all three versions of our analytical scheme have the advantage that there is no spatial discretization. Next let us consider the local iteration in the form of the third item (iii) above in more detail. At each time step l we consider maps of the form

$$\mathbf{f} \rightarrow \mathbf{v}^{r,f,\rho,l} = F_l^r(\mathbf{f}), \tag{93}$$

where $\mathbf{f} = (f_1, \cdots, f_n)^T$, and where $\mathbf{v}^{r,f,\rho,l} = \left(v_1^{r,f,\rho,l}, \cdots, v_n^{r,f,\rho,l}\right)^T$ satisfies the equation

$$\left\lbrace\begin{array}{ll}
\frac{\partial v_i^{r,f,\rho,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,f,\rho,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n f_j \frac{\partial v_i^{r,f,\rho,l}}{\partial x_j} = L_i^{\rho,l,f}(\mathbf{r}, \mathbf{v}) + \\
\qquad \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial f_k}{\partial x_j} \frac{\partial f_j}{\partial x_k}\right)(\tau, y) dy + r_{i,\tau}^l, \\
\\
\mathbf{v}^{r,f,\rho,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.).
\end{array}\right. \tag{94}$$

Here we assume that we have solved the Navier-Stokes equation in transformed coordinates for $\tau \leq l-1$ and the initial data at $l-1$ are given by

the solution at this time. The domain of the map F^r_l is

$$D_{F^r_l} := \left\{ \mathbf{f} \in \left[C^{1,2}_b\left([(l-1), l] \times \mathbb{R}^n \right) \right]^n | \mathbf{f}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.) \right\}. \tag{95}$$

In case $l=1$ we have $\mathbf{v}^{r,f,\rho,1}(0,.) = \mathbf{h}$, of course. With the appropriate choice of ρ_l and r^l_i a local scheme for $\mathbf{v}^{r,\rho,l}$ may be defined in terms of a functional series $(\mathbf{v}^{r,\rho,k,l})_k$ with $\lim_{k\uparrow\infty} \mathbf{v}^{r,\rho,k,l} = \mathbf{v}^{r,\rho,l}$. We start the iteration determing for $1 \leq i \leq n$ the functions ϕ^l_i and then the functions r^l_i (solving a certain equation as scetched in β above). Then for each $k \geq 0$ we have

$$\left\{ \begin{array}{l} \frac{\partial v^{r,\rho,k,l}_i}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v^{r,\rho,k,l}_i}{\partial x^2_j} + \rho_l \sum_{j=1}^n v^{r,\rho,k-1,l}_j \frac{\partial v^{r,\rho,k,l}_i}{\partial x_j} = L^{\rho,l}_i\left(\mathbf{r}^l, \mathbf{v}^{r,\rho,k-1,l} \right) + \\ \\ \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,m=1}^n \left(\frac{\partial v^{r,\rho,k-1,l}_m}{\partial x_j} \frac{\partial v^{r,\rho,k-1,l}_j}{\partial x_m} \right) (\tau, y) dy + r^l_{i,\tau}, \\ \\ \mathbf{v}^{r,\rho,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.), \end{array} \right. \tag{96}$$

along with $\mathbf{v}^{r,\rho,-1,l} = \mathbf{v}^{r,\rho,l-1}$. Let us look at the difference of two successive approximations $\mathbf{v}^{r,\rho,k-1,l}$ and $\mathbf{v}^{r,\rho,k,l}$ where we consider fixed functions $\mathbf{f}$ and $\mathbf{g}$ instead of $\mathbf{v}^{r,\rho,k-1,l}$ etc. as coefficient functions. Comparing $\mathbf{v}^{f,\rho,l}$ with $\mathbf{v}^{g,\rho,l}$ leads us to an expression for the difference which we denote by

$$\delta \mathbf{v}^{r,f,g,\rho,l} := \mathbf{v}^{r,f,\rho,l} - \mathbf{v}^{r,g,\rho,l}. \tag{97}$$

This function satisfies the equation

$$\left\{ \begin{array}{l} \frac{\partial \delta v^{r,f,g,\rho,l}_i}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \delta v^{r,f,g,\rho,l}_i}{\partial x^2_j} + \rho_l \sum_{j=1}^n f_j \frac{\partial v^{r,f,g,\rho,l}_i}{\partial x_j} \\ \\ \qquad = L^{\rho,l,f,g,0}_i\left(\mathbf{r}^l, \delta \mathbf{v}^{r,f,g,\rho,l} \right) + \\ \\ \qquad -\rho_l \sum_{j=1}^n \left(f_j - g_j \right) \frac{\partial v^{r,g,\rho,l}_i}{\partial x_j} \\ \\ \qquad +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,k=1}^n \left(\frac{\partial f_k}{\partial x_j} \frac{\partial f_j}{\partial x_k} \right) (\tau, y) dy, \\ \\ \qquad -\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,k=1}^n \left(\frac{\partial g_k}{\partial x_j} \frac{\partial g_j}{\partial x_k} \right) (\tau, y) dy, \\ \\ \delta \mathbf{v}^{r,f,g,\rho,l}(l-1,.) = 0. \end{array} \right. \tag{98}$$

Here we have

$$\begin{array}{l} L^{\rho,l,f,g,0}_i\left(\mathbf{r}^l, \delta \mathbf{v}^{r,f,g,\rho,l} \right) := \\ \\ L^{\rho,l,f,0}_i\left(\mathbf{r}^l, \mathbf{v}^{r,f,\rho,l} \right) - L^{\rho,l,g,0}_i\left(\mathbf{r}^l, \mathbf{v}^{r,g,\rho,l} \right) \end{array} \tag{99}$$

Note that the source term $r^l_{i,\tau}$ does not appear on the right side of this equation for the difference, furthermore all the terms involving only r^l_i and the derivatives of r^l_i (because they cancel out). Hence the first term $\mathbf{v}^{r,\rho,0,l}$ of the functional series $\mathbf{v}^{r,\rho,k,l}$ should contain the essential information concerning the growth of the solution at time step l and the higher order terms of the local iteration. In the first iterative time step of an iteration scheme related to (98) we shall have $f_j = v^{r,\rho,1,l}_j$ and $g_j = v^{r,\rho,0,l}_j$, such that the first term on the right side related to (98) becomes

$$-\rho_l \sum_{j=1}^n (f_j - g_j) \frac{\partial v^{r,g,\rho,l}_i}{\partial x_j} = -\rho_l \sum_{j=1}^n \left(v^{r,\rho,1,l}_j - v^{r,\rho,0,l}_j \right) \frac{\partial v^{r,\rho,0,l}_i}{\partial x_j}. \tag{100}$$

In [6] we considered a bound of $v^{\rho,0,l}_i$ and $v^{\rho,0,1}_{i,j}$ of form $C^* C_l$ where C_l depends linearly on l and C^* is a constant independent of the time step number l. Linear dependence of C_l of both terms with respect to l is sufficient in order to make our scheme global as the sum of time step sizes $\sum_{l\geq 1} \rho_l$ is unbounded. However, in this paper we shall construct a uniform bound $C_1 C^*$ for $v^{r,\rho,0,l}_{i,j}$ which is independent of the time step number l. Let us stick to this second term of right side of (98) for a moment. In the k-th iteration step the contribution to the first term on the right side related to (98) becomes

$$\begin{array}{l} -\rho_l \sum_{j=1}^n \left(v^{r,\rho,k,l}_j - v^{r,\rho,k-1,l}_j \right) \frac{\partial v^{r,\rho,k-1,l}_i}{\partial x_j} = \\ -\rho_l \sum_{j=1}^n \left(v^{r,\rho,k,l}_j - v^{r,\rho,k-1,l}_j \right) \left(\frac{\partial v^{r,\rho,0,l}_i}{\partial x_j} + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j} \delta v^{r,\rho,m,l}_i \right), \end{array} \tag{101}$$

such that we may estimate these terms by using a contraction property of the differences $\delta \mathbf{v}^{r,\rho,k,l}$ which we shall observe for the specific series

$$\begin{array}{l} \mathbf{v}^{r,\rho,k,l} = \mathbf{v}^{r,\rho,0,l} + \sum_{m=1}^k \delta \mathbf{v}^{r,\rho,m,l} = \\ \mathbf{v}^{r,\rho,0,l} + \sum_{m=1}^k \left(\mathbf{v}^{r,\rho,m,l} - \mathbf{v}^{r,\rho,m-1,l} \right). \end{array} \tag{102}$$

Now consider again the equation (98). Let us consider $r^l_i = 0$ for $1 \leq i \leq n$, since we have no source terms r^l_i in the higher correction terms for $\delta \mathbf{v}^{r,\rho,k,l}$ (in the proof of the main theorem below we shall see the additional terms $L^{\rho,l,f,g,0}_i \left(\mathbf{r}, \delta \mathbf{v}^{r,\rho,k,l} \right)$ with $\mathbf{f} = \mathbf{v}^{\rho,k,l}$ and $\mathbf{g} = \mathbf{v}^{\rho,k,l}$ do not alter the reasoning of time local convergence essentially). Note that for $r^l_i = 0$ we denote

$$\mathbf{v}^{\rho,k,l} = \mathbf{v}^{0,\rho,k,l} = \mathbf{v}^{r,\rho,k,l}, \tag{103}$$

and similar for all components $v^{\rho,k,l}_i$. Since $\mathbf{f} \in D_{F_l}$ we have that for each $1 \leq i \leq n$ the fundamental solution Γ^l_f of the scalar equation

$$\frac{\partial \Gamma^l_f}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \Gamma^l_f}{\partial x_j^2} + \rho_l \sum_{j=1}^n f_j \frac{\partial \Gamma^l_f}{\partial x_j} = 0 \tag{104}$$

exists. Similar, since $\mathbf{g} \in D_{F_l}$ we have that for each $1 \leq i \leq n$ the fundamental solution Γ^l_g of the scalar equation

$$\frac{\partial \Gamma^l_g}{\partial \tau} - \rho_l \nu \sum_{j=1}^{n} \frac{\partial^2 \Gamma^l_g}{\partial x_j^2} + \rho_l \sum_{j=1}^{n} g_j \frac{\partial \Gamma^l_g}{\partial x_j} = 0 \tag{105}$$

exists. Then formally we may represent the solution of the equation (94) in terms of the this fundamental solution (recall that we consider $\mathbf{r} = 0$ for simplicity). Since $\mathbf{v}^{f,\rho,l}$ and $\mathbf{v}^{g,\rho,l}$ have the same initial data we have for $1 \leq i \leq n$ (recall that $\mathbf{r} \equiv 0$ this time)

$$\begin{array}{l} v_i^{f,\rho,l}(\tau,x) - v_i^{g,\rho,l}(\tau,x) = \\ \\ -\int_0^\tau \int_{\mathbb{R}^n} \sum_{j=1}^n \left(f_j - g_j\right)(s,y) \frac{\partial v_i^{g,\rho,l}}{\partial x_j}(s,y) \Gamma^l_f(\tau,x;s,y) dy ds + \\ \\ \int_0^\tau \rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{n,i}(z-y) \times \\ \\ \left(\sum_{j,k=1}^n \left(f_{k,j} f_{j,k}\right)(s,y) - \sum_{j,k=1}^n \left(g_{k,j} g_{j,k}\right)(s,y)\right) \Gamma^l_f(\tau,x;s,z) dy dz ds, \end{array} \tag{106}$$

where we denote

$$f_{k,j} = \frac{\partial f_k}{\partial x_j} \quad \text{etc.} \tag{107}$$

for simplicity. (Note that in (106) we may express the first spatial derivatives of the function $\mathbf{v}^g$ in terms of an integral involving the respective first spatial derivatives of the fundamental solution Γ^l_g). In order to construct a fixed point we shall use some decay at spatial infinity. Indeed we need a certain decay at spatial infinity of the approximating functions $\mathbf{v}^{\rho,k,l}$ of $\mathbf{v}^{f,\rho,l}$ in order to estimate the limit of the functional series $\left(\mathbf{v}^{\rho,k,l}\right)_k$ (with respect to some appropriate norm, for example a Sobolev norm $|.|_{H^s}$ for $s \geq \frac{n}{2} + \alpha$. In order to construct an iteration scheme for the higher order corrections at each time step l, i.e., the functions $\delta \mathbf{v}^{r,\rho,k,l}$ for $k \geq 1$, we observe that

$$\begin{array}{l} \sum_{j,k=1}^n \left(f_{k,j} f_{j,k}\right)(s,y) - \sum_{j,k=1}^n \left(g_{k,j} g_{j,k}\right)(s,y) = \\ \\ \sum_{j,k=1}^n \left(f_{k,j} f_{j,k}\right)(s,y) - \sum_{j,k=1}^n \left(f_{k,j} g_{j,k}\right)(s,y) + \\ \\ \sum_{j,k=1}^n \left(g_{k,j} f_{j,k}\right)(s,y) - \sum_{j,k=1}^n \left(g_{k,j} g_{j,k}\right)(s,y) = \\ \\ \left(\sum_{j,k=1}^n \left(f_{k,j}(s,y) + g_{k,j}(s,y)\right)\right) \times \\ \\ \left(\sum_{j,k=1}^n \left(f_{j,k}(s,y) - g_{j,k}(s,y)\right)\right). \end{array} \tag{108}$$

In order to deal with the problem of constructing a fixed point in a function space of infinite domain we use the standard assumption concerning

decay of regular initial data at spatial infinity, i.e. we assume that the map $x \to \mathbf{h}(x) = (h_1(x), \cdots, h_n(x))$ is a given function with components h_i in $C^\infty(\mathbb{R}^n) \cap H^s$ for all $1 \leq i \leq n$ and s large enough. Here, we write $H^s = H^s(\mathbb{R}^n)$. Now three different procedures are possible in order to construct a fixed point for each time step l in an appropriate function space, i.e. locally in time. The first method a) below makes use of embedding theorems for convergence of local schemes. It is remarkable that Hilbert space theory suffices in order to deal with the most interesting case of dimension $n = 3$. The following method can be generalized in order to include arbitrary dimensions by considering embedding theorems for spaces of Hölder type (so called Zygmund spaces C^s_*). These spaces coincide with classical Hölder spaces for noninteger values s. We have the standard embedding theorem

Theorem 1.3. *For all $s \in \mathbb{R}^n$ and $p \in (1, \infty)$*

$$H^{s,p}(\mathbb{R}^n) \subset C^r_*(\mathbb{R}^n) \tag{109}$$

for $r = s - \frac{n}{p}$

Here the spaces $H^{s,p}$ are defined similarly as in the case $p = 2$ and coincide with the spaces $H^{k,p}$ for integers k which have weak derivatives up to order k in L^p. We do not repeat the definitions here (which can be looked up in standard textbooks) since this of marginal importance for us, i.e., in order to remark that the following method a) can be realized for arbitrary dimension $n \geq 1$. We shall see later that the integral magnitude in (12) is finite.

a) The following method can applied in the case of arbitrary dimension (cf. preceding remarks), but for simplicity we consider the case $n = 3$. This is a construction in $\left[C^{1,2}_b([l-1, l] \times \mathbb{R}^n)\right]^n \cap \left[H^2_l\right]^n$, where

$$H^2_l := \left\{ f \in C_b([l-1, l] \times \mathbb{R}^n) \,|\, f(t, .) \in H^2(\mathbb{R}^n),\ \forall\, t \in [l-1, l] \right\}. \tag{110}$$

This is the space of vector-valued functions $\mathbf{h} = (h_1, \cdots, h_n)^T$ with $h_i \in C^{1,2}_b$, i.e., h_i is in $C^{1,2}$ with bounded derivatives of first order with respect to time and second order with respect to space. For the series $\mathbf{v}^{\rho,k,l}$ defined in (iii) above we can establish that $\mathbf{v}^{\rho,k,l} \in \left[C^{1,2}_b([l-1, l] \times \mathbb{R}^n)\right]^n$ such that for some $0 < c < 1$ we have

$$|\mathbf{v}^{r,\rho,k+1,l} - \mathbf{v}^{r,\rho,k,l}|^n_{1,2} \leq c|\mathbf{v}^{r,\rho,k,l} - \mathbf{v}^{r,\rho,k-1,l}|^n_{1,2}, \tag{111}$$

where

$$|\mathbf{f}|^n_{1,2} := \sum_{i=1}^n \left[|f_i|_0 + \sum_{j=1}^n |f_{i,j}|_0 + \sum_{j,k=1}^n |f_{i,j,k}|_0 \right], \tag{112}$$

and where $|.|_0$ denotes the supremums norm. Then we can show that

$$\mathbf{f} \in \left[H^2_l\right]^n \to \mathbf{v}^f \in \left[H^2_l\right]^n. \tag{113}$$

Then our scheme leads to a series $\left(\mathbf{v}^{\rho,k,l}\right)_k \in \left[C^{1,2}_b\left([l-1,l]\times\mathbb{R}^n\right)\right]^n \cap \left[H^2_l\right]^n$ with a limit

$$\mathbf{v}^{r,\rho,l} \in \left[H^2_l\right]^n. \tag{114}$$

Since $n = 3$ the functions $v^{\rho,l}_i$ are Hölder continuous with respect to the spatial variable (uniformly in time τ) and the fundamental solution $\Gamma^{r,l}_v$ of

$$\frac{\partial \Gamma^{r,l}_v}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \Gamma^{r,l}_v}{\partial x^2_j} + \rho_l \sum_{j=1}^n v^{\rho,l}_j \frac{\partial \Gamma^{r,l}_v}{\partial x_j} - L^{\rho,l,0}_i(\mathbf{r},\mathbf{v}) = 0 \tag{115}$$

exists. Then we can use the representation

$$\begin{array}{l}
v^{r,\rho,l}_i(r,\tau,x) = \int_{\mathbb{R}^n} h_i(y)\Gamma^{r,l}_v(\tau,x;0,y)dy+ \\
\int_0^\tau \rho_l \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_{n,i}(z-y)\times \\
\sum_{j,k=1}^n \left(v^{r,\rho,l}_{k,j} v^{r,\rho,l}_{j,k}\right)(s,y)\Gamma^{r,l}_v(\tau,x;s,z)dydzds \\
-\int_0^\tau \int_{\mathbb{R}^n} r^l_{i,\tau}(s,y)\Gamma^{r,l}_v(\tau,x;s,z)dydzds, \\
+\int_0^\tau \int_{\mathbb{R}^n} L^{\rho,l,0}_i(\mathbf{r},\mathbf{v})(s,y)\Gamma^{r,l}_v(\tau,x;s,z)dydzds,
\end{array} \tag{116}$$

where the existence of the first spatial derivatives on right side of (116) may be established directly. Existence of second order spatial derivatives and first order time derivatives may be derived using partial integration.

Remark 1.4. In a subsequent paper concerning Navier-Stokes equations on manifolds we shall show that from a closer look at (111) we may even conclude that for some $c \in (0,1)$ we have a time-local contraction with respect to Hölder norms, i.e.,

$$|\mathbf{v}^{r,\rho,k+1,l} - \mathbf{v}^{\rho,k,l}|^n_{\alpha/2,2+\alpha}| \leq c|\mathbf{v}^{r,\rho,k,l} - \mathbf{v}^{\rho,k-1,l}|^n_{\alpha/2,2+\alpha}. \tag{117}$$

For compact mainfolds this implies that we have a local contraction on a Banach space, and this simplifies the analysis. Note, however, that the usual the introduction of another time transformation which introduces a potential term of a specific sign is critical. A naive use of this Schauder type estimates within our scheme does not lead to a global scheme. Here we use standard notation for Hölder norms,

where the first subscript $\alpha/2$ refers to the modulus of Hölder continuity with respect to the time variable and the second subscript refers to the modulus of Hölder continuity of the (second derivatives) of the functions $v^{r,\rho,k,l}$ with respect to the spatial variables. More precisely, define the Euclidean distance in $\mathbb{R}^{n+1}$ between the points $y_1 = (t_1, x_1), y_2 = (t_2, x_2)$ by

$$e(z_1, z_2) = \sqrt{|t_1 - t_2|} + |x_1 - x_2|. \tag{118}$$

If w is a function in a domain $D \subset \mathbb{R}^{n+1}$ we denote for $\alpha \in (0, 1)$

$$[w]_{\alpha/2,\alpha,D} = \sup_{y_1 \neq y_2; y_1, y_2 \in D} \frac{|w(y_1) - w(y_2)|}{e^{\alpha}(y_1, y_2)}. \tag{119}$$

Next define

$$|w|_{\alpha/2,\alpha;D} = |w|_{0,D} + [w]_{\alpha/2,\alpha;D\cdot}, \tag{120}$$

Similarly, we define

$$[w]_{\alpha/2,1+\alpha,D} = \sum_{j=1}^{n} [w_{,j}]_{\alpha/2,\alpha,D}, \tag{121}$$

and with the notation $w_t := \frac{\partial w}{\partial t}$ we have

$$[w]_{1+\alpha/2,2+\alpha,D} = \sum_{j=1}^{n} [w_t]_{\alpha/2,\alpha,D} + \sum_{j,k=1}^{n} [w_{,j,k}]_{\alpha/2,\alpha,D}. \tag{122}$$

This leads to the notation of more regular HöDer spaces

$$|w|_{\alpha/2,1+\alpha;D} := |w|_{0,1;D} + [w]_{\alpha/2,1+\alpha,D} \tag{123}$$

and

$$|w|_{1+\alpha/2,2+\alpha;D} := |w|_{1,2;D} + [w]_{1+\alpha/2,2+\alpha,D}, \tag{124}$$

where we use the notation

$$|w|_{0,1;D} := |w|_{0;D} + \sum_{i=1}^{n} |w_{x_i}|_{0;D}, \text{ and} \tag{125}$$

$$|w|_{0,2;D} := |w|_{0;D} + \sum_{i=1}^{n} |w_{x_i}|_{0;D} + + \sum_{i,j=1}^{n} |w_{x_i x_j}|_{0;D}, \tag{126}$$

and

$$|w|_{1,2;D} := |w|_{0;D} + \sum_{i=1}^{n} |w_{x_i}|_{0;D} + |w_t|_{0;D} + \sum_{i,j=1}^{n} |w_{x_i x_j}|_{0;D}. \tag{127}$$

Note that the latter norms do not define Banach spaces. However, below we shall use the fact that a functional series which is uniformly and absolutely bounded with uniformly and absolutely bounded derivatives can be differentiated term by term. For this matter (127) is useful. If the domain D is determined from the context we shall suppress it in notation, especially if D is of the form $[S,T]\times\mathbb{R}^n$.

b) Another possibility (a more elementary way) is the following. We use the relation (111) and apply it iteratively starting with the function $\mathbf{v}^{\rho,l-1}$ and get a series $\left(\mathbf{v}^{r,\rho,k,l}\right)_k \in C^{1,2}_b$ with

$$\mathbf{v}^{\rho,k,l}=\mathbf{v}^{r,\rho,l-1}+\sum_{m=0}^{k}\delta\mathbf{v}^{r,\rho,m,l}\in\left[C^{1,2}_b\left([l-1,l]\times\mathbb{R}^n\right)\right]^n, \tag{128}$$

where

$$\delta\mathbf{v}^{r,\rho,m,l}=\mathbf{v}^{r,\rho,m,l}-\mathbf{v}^{r,\rho,m-1,l}. \tag{129}$$

Next assume that for all $x\in\mathbb{R}^n$ and for multiindices α with $|\alpha|\leq 2$ and $k\leq 5$ we have constants $C_{\alpha k}$ such that

$$|\partial^\alpha_x\mathbf{h}(x)|\leq\frac{C_{\alpha k}}{(1+|x|)^k} \tag{130}$$

For each substep k of each timestep l we shall show that for $|\alpha|\leq 2$ the approximations $\mathbf{v}^{r,\rho,k,l}$ satisfy

$$|\partial^\alpha_x\mathbf{v}^{r,\rho,k,l}(t,x)|\leq\frac{C_{\alpha k}}{(1+|x|)^5}\text{ if }|\alpha|\leq 2\qquad . \tag{131}$$

Then the decay relation for the solution helps us to control convergence of our scheme by consideration of convergence of a functional series in transformed spatial coordinates. In transformed spatial coordinates we may use

Proposition 1.5. *Let K be a compact domain and assume that a functional series $(h_k)_k : C_b(K)\rightarrow\mathbb{R}$ is given. Assume that for all $x\in K$ the series $\sum_{k\in\mathbb{N}}h_k(x_0)$ converges, and assume that for $1\leq j\leq n$ the series $\sum_{k\in\mathbb{N}}h_{k,j}$ converges uniformly in K. Then the series $\sum_{k\in\mathbb{N}}h_k$ converges uniformly to a function h^* which is differentiable and such that for $1\leq j\leq n$ we have*

$$h_{,j}(x)=\sum_{k\in\mathbb{N}}h_{k,j}(x). \tag{132}$$

The transformation uses the polynomial decay of order 5 described above. Application of this lemma then leads to a classical solution $\mathbf{w}^l$ (locally in time) on a compact space K_l (the image of a coordinate transformation of D^τ_l) and then to a classical solution v^l on D^τ_l.

c) The third possibility is to do the contraction estimate in a stronger Hölder norm, i.e.

$$|\mathbf{v}^{r,\rho,k+1,l}-\mathbf{v}^{r,\rho,k,l}|^n_{1+\alpha/2,2+\alpha}|\leq c|\mathbf{v}^{r,\rho,k,l}-\mathbf{v}^{r,\rho,k-1,l}|^n_{1+\alpha/2,2+\alpha}. \tag{133}$$

and then establish decay at spatial infinity as in b). Then in transformed coordinates for the equivalent series $\left(\mathbf{w}^{r,\rho,k,l}\right)_k$ we have $\mathbf{w}^{r,\rho,k,l}\in C_{1+\alpha/2,2+\alpha}\left(K_l\right)$, and $\mathbf{w}^{r,\rho,l}\in C_{1+\alpha/2,2+\alpha}\left(K_l\right)$ as $k\uparrow\infty$. Note that the latter space is a Banach space.

We may summarize our construction as follows. A solution of the Navier Stokes equation system which is given by a global fixed point

$$\mathbf{f}\rightarrow\mathbf{v}^f=F(\mathbf{f}), \tag{134}$$

where $\mathbf{f}=(f_1,\cdots,f_n)^T$ is defined on $[0,\infty)\times\mathbb{R}^n$, and $\mathbf{v}^f$ satisfies the equation

$$\left\lbrace\begin{array}{ll}\frac{\partial v^f_i}{\partial t}-\nu\sum_{j=1}^n\frac{\partial^2 v^f_i}{\partial x_j^2}+\sum_{j=1}^n f_j\frac{\partial v^f_i}{\partial x_j}=\\ \\ \qquad\qquad\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial f_k}{\partial x_j}\frac{\partial f_j}{\partial x_k}\right)(\tau,y)dy,\\ \\ \mathbf{v}^f(0,.)=\mathbf{h}(0,.).\end{array}\right. \tag{135}$$

This fixed point may be denoted by

$$\mathbf{v}^*:=F(\mathbf{v}^*)=\mathbf{v}^{\mathbf{v}^*}. \tag{136}$$

It seems difficult to construct this fixed point directly. Therefore, we construct a fixed point of an equivalent problem which has the solution $\mathbf{v}^r=\mathbf{v}+\mathbf{r}$ in successive time steps l on the domains $[l-1,l]\times\mathbb{R}^n$, where $\mathbf{r}=(r_1,\cdots,r_n)$ is a bounded function. We solve for $\mathbf{v}^r$ in time steps with functions $\mathbf{v}^{r,\rho,l}=\left(v_1^{r,\rho,l},\cdots,v_n^{r,\rho,l}\right)^T$, via a functional series

$$\mathbf{v}^{r,\rho,l}=\mathbf{v}^{r,\rho,0,l}+\sum_{k\geq 1}\delta\mathbf{v}^{r,\rho,k,l}. \tag{137}$$

We establish a contraction property for the specific functional series $\mathbf{v}^{r,\rho,k,l}$ and prove convergence to $\mathbf{v}^{r,\rho,l}$. The function $\mathbf{r}$ equals $\mathbf{r}^l$ on each domain $(l-1,l]\times\mathbb{R}^n$, and is defined recursively in such a way that the solution $\mathbf{v}^r$ is bounded and $\mathbf{r}$ bounded globally in time. Here, $\mathbf{r}^l$ solves at each time step a linearized equation of Navier-Stokes type with 'consumption' source term. Note that this is crucial for global convergence. The local

contraction property for the higher order corrections follow from a priori estimates of fundamental solutions of approximative subproblems. We shall use the Levy expansion of scalar linear fundamental solutions. We may also use the following standard result.

Remark 1.6. Note that the application of standard Schauder estimates (as will be used in the second part of this article) requires a certain sign of the coefficient of the potential term. This sign is not given for the successive problems for $\mathbf{v}^{r,\rho,l}$, $l \geq 1$. However, since the functions r_i^l are bounded, we can always consider equivalent problems related to

$$u_i^{r,\rho,l,\lambda}(\tau,x) = e^{-\lambda\tau} v_i^{r,\rho,l}(\tau,x) \tag{138}$$

with $\lambda > 0$ large enough have coefficients for potential terms with the same sign as the time derivative.

This is also a difference to our construction in [6] for the multidimensional Burgers equation. If the transformation (138) and standard Schauder estimates are used in order to establish a local contraction property of our scheme, then it is not sufficient to prove linear growth of the solution globally. However, since we prove he boundedness of the solution we can even use standard Schauder estimates at each time step in order to prove local convergence. For the sake of completeness let us refer to this standard results which may be found in [11].

Let L be a scalar parabolic partial differential operator defined by

$$\begin{array}{l} Lf(t,x) \equiv \frac{\partial}{\partial t} f(t,x) - \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial^2}{\partial x_i \partial x_j} f(t,x) \\ + \sum_{i=1}^{n} b_i(t,x) \frac{\partial}{\partial x_i} f(t,x) + c(t,x) f(t,x) \end{array} \tag{139}$$

for all $f \in C^{1,2}(Q_T)$ for arbitrary $T \in (0,\infty)$.

Remark 1.7. Note that any potential term cf has the same (positive) sign as the time derivative.

Let us assume that the spatial part of the operator L is uniformly elliptic with ellipticity constant $\lambda \in (0,\infty)$ (ellipticity from below) and $\Lambda \in (\lambda,\infty)$ (ellipticity from above), i.e. for all

$$\forall\, \xi \in \mathbb{R}^n \setminus \{0\}\,, \quad \xi \in \mathbb{R}^n:\; \lambda|\xi|^2 \leq \sum_{ij} a_{ij}(t,x)\xi_i\xi_j \leq \Lambda|\xi|^2 \tag{140}$$

Furthermore we assume that the coefficients have bounded Hölder norms, i.e.

$$|a_{ij}|_{\delta/2,\delta,Q_T} \leq K,\; |b_i|_{\delta/2,\delta,Q_T} \leq K,\; |c|_{\delta/2,\delta,Q_T} \leq K \tag{141}$$

for some constant $K > 0$.

Theorem 1.8. *Assume that (140) and (141) hold and that* $c \leq r$ *for some constant* $r > 0$ *(note that minus sign in front of the second order coefficients in the definition of the parabolic operator* L*, cf. remark (1.7)). Let* $g \in C^{2+\delta}(\mathbb{R}^n)$ *and* $f \in C^{\delta/2,\delta}(Q_T)$*. Then there exists a unique function* $u \in C^{1+\delta/2,2+\delta}(Q_T)$ *which solves the Cauchy problem*

$$\begin{cases} Lu = f \ \text{ on } Q_T, \\ \\ u(0,.) = g(.) \ \text{ on } \mathbb{R}^n. \end{cases} \tag{142}$$

Moreover the estimate

$$|u|_{1+\delta/2,2+\delta,Q_T} \leq C\left(|g|_{2+\delta,\mathbb{R}^n} + |f|_{\delta/2,\delta,Q_T}\right) \tag{143}$$

holds for some constant $C > 0$ *depending only on* $n, \lambda, \Lambda, \delta, K$.

If we want to apply the latter estimate then it is not enough to show that the growth of $\mathbf{v}^{r,\rho,l}$ is at most linear. This is even more obvious as in the case where we deal with the weaker norm $|.|_{1,2}$. However, using the stronger Hölder norm leads to an additional difficulty with respect to the control of the integral magnitude (12). Instead, we shall show constructively that we have a classical bounded solution with an integral magnitude (12) which has linear growth with respect to time. This is achieved by a dynamical choice of the functions r_i^l, i.e., a recursive construction with respect to the time step number l. In principal this allows us to apply the Schauder type estimates locally with respect to time. However, in this paper we use classical expansions of fundamental solutions. In case of the Navier-Stokes equation on compact manifolds (the flat n-torus is a simple example of a flat one) we do not have to deal with the integral norm seperately. However, some different difficulties arise if the coefficient of second order depend on the spatial variables. We shall treat this case in a subsequent paper.

In order to prove the existence of a bounded solution $\mathbf{v} = (v_1, \cdots, v_n)$ of the incompressible Navier Stokes equation in its Leray projection form we construct a bounded function $\mathbf{r} = (r_1, \cdots, r_n)$ which is smooth with respect to the spatial variables and differentiable with respect to time except at a discrete set of points. Moreover the functions $\tau \to r_i(\tau, x)$ are in $H^{1,p}$ for all $1 \leq i \leq n$ and $p \geq 1$, i.e. the first time derivative is in L^p for $p \geq 1$. The values where the time derivatives of $\tau \to r_i(\tau, x)$ do not exist in a classical sense coincide with the time discretizaion of our scheme, i.e. with the values of natural numbers ($\tau \in \mathbb{N}$). The bounded function $\mathbf{r}$ is constructed recursively at each time step l. Hence $\mathbf{r}$ is determined by a sequence of functions $\mathbf{r}^l = (r_1^l, \cdots, r_n^l)$, where each $\mathbf{r}^l$ is defined on the domain $[l-1, l] \times \mathbb{R}^n$. Instead of solving the incompressible Navier-Stokes equation we solve a corresponding equation for the function

$$(\tau, x) \to \mathbf{v}^{r,\rho} := \mathbf{v}^\rho + \mathbf{r}, \tag{144}$$

time step by time step, i.e., in the form $\mathbf{v}^{r,\rho,l} = \mathbf{v}^{\rho,l} + \mathbf{r}^l$ for $l \geq 1$ where each $\mathbf{v}^{r,\rho,l}$ is defined on the domain $[l-1,l]\times\mathbb{R}^n$. We first determine $\mathbf{r}^l$ from data of the previous time step, i.e., from $\mathbf{v}^{r,\rho,l-1}$ (or from the initial data $\mathbf{h}$ if $l=1$. This is done in such a way that the locally constructed solution for $\mathbf{v}^{r,\rho,l}$ stays bounded independently of the time step number l.

This is the outline of this paper. In the next Section 2 we prove the main theorem, i.e., the existence of a bounded regular function $\mathbf{r}$ such that $\mathbf{v}^r := \mathbf{v}+\mathbf{r}$ satisfies an equation equivalent to the incompressible Navier-Stokes equation in an (almost) classical sense. This leads to a classical solution $\mathbf{v}$ for the Navier-Stokes equation itself. Then in Section 3 we draw conclusions concerning regularity, uniqueness, and extensions to equations with external forces. Finally, in Section 4 we sketch an algorithm for the solution of the Navier-Stokes equation by iterative use of local analytic expansions of scalar parabolic equations. This algorithm simplifies the construction of the global solution via the function $\mathbf{v}^r$ since we can use the information that the solution is bounded.

2 Main theorem

We shall assume that the initial data function $x \rightarrow \mathbf{h}(x) = (h_1(x),\cdots,h_n(x))^T$ with components h_i in $C^\infty(\mathbb{R}^n)$ for all $1\leq i\leq n$ satisfies

$$|\partial_x^\alpha \mathbf{h}(x)| \leq \frac{C_{\alpha k}}{(1+|x|)^k} \tag{145}$$

for all $x\in\mathbb{R}^n$, and for all multiindices α and and integers k with some constants $C_{\alpha k}$.

Remark 2.1. If there is an external force term $\mathbf{f}_{ex}$ included on the right then we may assume that

$$|\partial_t^m\partial_x^\alpha \mathbf{f}_{ex}(t,x)| \leq \frac{C_{\alpha m k}}{(1+|x|+t)^k} \qquad \text{for all} \qquad (t,x)\in[0,\infty)\times\mathbb{R}^n, \tag{146}$$

for any multiindices α, integers k, and nonnegative integers m, and with some constants $C_{\alpha m k}$.

Essentially, the assumptions on the external forces $\mathbf{f}_{ex}$ and the initial data $\mathbf{h}$ mean that these functions are located in Sobolev spaces H^s of arbitrary order $s\in\mathbb{R}$, i.e., for all $s\in\mathbb{R}$ we have

$$\mathbf{h}\in\left[H^s\left(\mathbf{R}^n\right)\right]^n, \tag{147}$$

and for all $t\in[0,\infty)$

$$\mathbf{f}_{ex}(t,.)\in\left[H^s\left(\mathbf{R}^n\right)\right]^n. \tag{148}$$

Here, for $s\in\mathbb{R}$ H^s is an ordinary Hilbert space with

$$H^s\equiv H^s\left(\mathbb{R}^n\right) = \left\{ f\in\mathcal{S}'\left(\mathbb{R}^n\right) | \hat{f}(\xi)^2\left(1+|\xi|^2\right)^s d\xi<\infty\right\}, \tag{149}$$

where $\hat{f}$ denotes the Fourier transform and $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions on $\mathbb{R}^n$. In order to keep notation simple we consider the case $f_i = 0$ from now on (however, this is not essential and it will be clear that our construction can be generalized immediately to the case involving source terms satisfying (148) or (184)). As indicated in the introduction our proof of a bounded regular solution of the incompressible Navier-Stokes equation consists of three main ideas: a) we introduce a time discretization and a series of linear time transformations $t = \rho_l \tau$ such that time step size 1 in τ-coordinates is related to a small time step size in original coordinates and small coefficients of spatial derivatives in transformed time coordinates such that an iteration procedure leads to a local solution in time; b) in order to control the growth of the solution we introduce a locally regular function $\mathbf{r}$ and solve an equivalent problem for the function $\mathbf{v}^r := \mathbf{v} + \mathbf{r}$ at each time step where $\mathbf{r}$ is itself a bounded function. The function $\mathbf{r}$ solves a linearized Navier-Stokes type equation, but with a 'consumption' source term ϕ^l which has been explained in the introduction to some extent. This source term consists of n components $\phi^l = \left(\phi^l_1, \cdots, \phi^l_n\right)$ such that the functions $\mathbf{v}^{r,\rho,l}$ and $\mathbf{r}^l$ are bounded independent of the time step number l. Here, the 'consumption' function ϕ^l and the function $\mathbf{r}^l$ are defined recursively on the domains D^τ_l of each time step number l and depend on the solution $\mathbf{v}^{r,l-1}$ and $\mathbf{r}^{l-1}$ defined at the previous time step or dependent on the data $\mathbf{h}$ if $l = 1$; c) At each time step we ensure that the solution has some decay at spatial infinity which is necessary in order to prove the existence of a time local fixed point. In the case $n = 3$ it is sufficient to have $v^{r,\rho,l}_i(\tau, .) \in H^2$ for all $l \geq 1$ and $\tau \in [l-1, l]$.

Remark 2.2. In the introduction of this paper we mentioned that global regular solutions of the multivariate Burgers equation can be obtained by a priori estimates of the form (3). If we consider the method (i) described above and start at each time step with the solution of the associated Burgers equation with a certain source term then we need to prove that solutions of the associated Burgers equation satisfy a certain decay at spatial infinity (in case the initial condition at each time step satisfies a certain decay condition at spatial infinity). This is true in any case a), b), and c) of local fixed point construction above. For example, if we consider the method c) of the fixed point construction, then a decay of polynomial order 5 at spatial infinity is sufficient in order to prove the global convergence of our scheme to a classical solution. Hence, in order to apply method i) for the local iteration part it is sufficient to prove the following:

Proposition 2.3. *The Cauchy problem for the multivariate Burgers equation with initial data* $\mathbf{h} \in H^s$ *for* $s \in \mathbb{R}$ *has a classical solution* $\mathbf{u} \in \left[C_{1+\alpha/2,2+\alpha}\left([0,\infty) \times \mathbb{R}^n\right)\right)^n$ *such that* $\mathbf{u}(t,.)$ *satisfies (183) for* $k \leq 5$*, i.e.*

we have for all $t \in [0,\infty)$

$$|\partial_x^\alpha \mathbf{u}(t,.)| \leq \frac{C_{\alpha k}}{(1+|x|)^k} \text{ for } k \leq 5 \text{ and } |\alpha| \leq 2. \tag{150}$$

More over the solution is bounded by the initial data, i.e., (2) holds.

The proof of this proposition essentially follows from our considerations in [6] and from estimates of type (3). The result in Proposition 1 can be sharpened in order to construct more regular solutions. In this Section we deal with the construction of classical solutions (which is essential). However, since we use the alternative (iii) in this paper we do not use this proposition in the present proof.

Next let us consider the construction of the function $\mathbf{r}$ more closely. Assume that we have solved for $\mathbf{v}^{r,\rho,m} := \mathbf{v}^{\rho,m} + \mathbf{r}^m$ for $1 \leq m \leq l-1$ such that the solution for $\mathbf{v}^{r,\rho}$ is constructed on the domain $[0,l-1]\times\mathbb{R}^n$. Especially the functions $\mathbf{r}^m$ for $1 \leq m \leq l-1$ have been constructed. Consider the restriction $\mathbf{v}^{r,\rho,l-1}$ of $\mathbf{v}^{r,\rho}$ to the domain $[l-1,l]\times\mathbb{R}^n$. Equation (24) may be written explicitly in the form

$$\left\{\begin{array}{l} \frac{\partial v_i^{r,\rho,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l} \frac{\partial v_i^{r,\rho,l}}{\partial x_j} = \psi_i^l, \\ \\ \mathbf{v}^{r,\rho,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.), \end{array}\right. \tag{151}$$

where

$$\begin{array}{ll} \psi_i^l = L_i^{\rho,l}\left(\mathbf{r}^l, \mathbf{v}^{r,\rho,l}\right) & \\ & (152) \\ +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,l}}{\partial x_j} \frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy + r_{i,\tau}^l & \\ \\ = r_{i,\tau}^l - \rho_l \nu \Delta r_i^l + \rho_l \sum_{j=1}^n r_j^l \frac{\partial r_i^l}{\partial x_j} & \\ \\ -\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial r_k^l}{\partial x_j} \frac{\partial r_j^l}{\partial x_k}\right)(\tau,y)dy & \\ \\ +\rho_l \sum_{j=1}^n r_j^{\rho,l-1} \frac{\partial v_i^{r,\rho,l-1}}{\partial x_j} + \rho_l \sum_{j=1}^n v_j^{r,\rho,l-1} \frac{\partial r_i^{l-1}}{\partial x_j} & (153) \\ \\ -2\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial r_k^{l-1}}{\partial x_j} \frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,y)dy & \\ \\ +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{r,\rho,l}}{\partial x_j} \frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy. & \end{array}$$

In order to control the growth of the functions $\mathbf{v}^{r,\rho,l}$ at time step l we control 1) the growth of $\mathbf{v}^{r,\rho,0,l}$ which satisfies a linearized equation, and 2) ensure

that the correction $\mathbf{v}^{r\rho,l}-\mathbf{v}^{r,\rho,0,l}=\sum_{k=1}^{\infty}\delta\mathbf{v}^{r,\rho,k,l}$ is small enough by choosing ρ_l appropriately. Let us look at the linear approximation $\mathbf{v}^{r,\rho,0,l}$ of the function $\mathbf{v}^{r,\rho,l}$ first. The equation for $\mathbf{v}^{r,\rho,0,l}$ is

$$\begin{cases}\frac{\partial v_i^{r,\rho,0,l}}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}(l-1,.)\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j}=\\ L_i^{\rho,l}\left(\mathbf{r}^l,\mathbf{v}^{r,\rho,l-1}(l-1,.)\right)+r^l_{i,\tau}\\ +\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(l-1,y)dy,\\ \mathbf{v}^{r,\rho,0,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.).\end{cases}\tag{154}$$

Note that all functions with index $l-1$ are defined on the domain $[l-2,l-1]\times\mathbb{R}^n$ of the previous time step. Therefore these functions are alway evaluated at time $l-1$ if they occur in the lth time step. Considering the linearized equation for $\mathbf{v}^{r,\rho,0,l}$ has the advantage that we may represent the solution of (154) in terms of the fundamental solution of the equation

$$\frac{\partial v_i^{r,\rho,0,l}}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}(l-1,.)\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j}=0.\tag{155}$$

This representation involves a spatial integral with the initial data and a source term integral with respect to space and time involving the right side of the equation (154). The classical representation involves an integral with initial data which is solution of an equation where we can apply the maximum principle. The second term can be controlled by choice of $\mathbf{r}^l$ as indicated in the introduction. Next determining $\mathbf{r}^l$ in a straightforward way such that the right side of (154) becomes a function $\psi_i^{l,0}$ approximating the function ψ_i^l (151) would lead us to a Navier-Stokes type equation for $\mathbf{r}^l$. The equation is

$$\begin{cases}r^l_{i,\tau}-\rho_l\nu\Delta r^l_i+\rho_l\sum_{j=1}^n r^l_j\frac{\partial r^l_i}{\partial x_j}\\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial r^l_j}{\partial x_k}\right)(\tau,y)dy\\ +L_i^{\rho,l,0}\left(\mathbf{r}^l,\mathbf{v}^{r,\rho,l-1}\right)\\ +\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(l-1,y)dy\\ =\psi_i^{l,0}\\ \mathbf{r}^l(l-1,.)=\mathbf{r}^{l-1}(l-1,.).\end{cases}\tag{156}$$

However, solving for $\mathbf{r}^l$ in this form would imply that we have transferred the original difficulties to control the growth of the solution to $\mathbf{r}^l$. Instead, we consider a different right side ϕ^l_i which approximates $\psi^{l,0}_i$ for ρ_l small. Here, 'approximation' is in the sense that $\psi^{l,0}_i = \phi^l_i + \left(\psi^{l,0}_i - \phi^l_i\right)$, and the difference becomes small if ρ_l becomes small (cf. proof of main theorem below). Furthermore $\psi^{l,0}_i$ approximates ψ^l_i. Here, we let r^l_i solve a linearization of (151), i.e. the function $\mathbf{r}^l$ solves

$$\begin{cases} r^l_{i,\tau} - \rho_l\nu\Delta r^l_i + \rho_l\sum_{j=1}^n r^{l-1}_j(l-1,.)\frac{\partial r^l_i}{\partial x_j} = \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial r^{l-1}_j}{\partial x_k}\right)(l-1,y)dy \\ +L^{\rho,l,0}_i(\mathbf{r}^{l-1}(l-1,.);\mathbf{v}^{r,\rho,l-1}(l-1,.)) \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v^{r,\rho,l-1}_m}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_m}\right)(l-1,y)dy + \phi^l_i, \\ \mathbf{r}^l(l-1,.) = \mathbf{r}^{l-1}(l-1,.). \end{cases} \tag{157}$$

Note that it is the same $\mathbf{r}^l$ which solves this linearized equation which is the equation with a different source term ϕ^l_i. Hence the difference of $\psi^{l,0}_i$ and ϕ^l_i may be obtained by subtracting equation (157) from equation (156). We get

$$\begin{array}{l} \rho_l\sum_{j=1}^n(r^l_j - r^{l-1}_j)\frac{\partial r^l_i}{\partial x_j} \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial r^l_j}{\partial x_k}\right)(\tau,y)dy \\ +\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial r^{l-1}_j}{\partial x_k}\right)(l-1,y)dy \\ +L^{\rho,l,0}_i\left(\mathbf{r}^l - \mathbf{r}^{l-1},\mathbf{v}^{r,\rho,l-1}\right) \\ = \psi^{l,0}_i - \phi^l_i. \end{array} \tag{158}$$

We shall indeed see that we can make the right side of (158) small independently of l with the right choice of ρ_l and $\mathbf{r}^l$. How small do we want to get it? Well, we want to have ϕ^l_i close to $\psi^{l,0}_i$ and $\psi^{l,0}_i$ close to ψ^l_i where we want to choose ϕ^l_i such that it controls both, $\mathbf{v}^{r,\rho,l}$ and $\mathbf{r}^l$. This is done as follows: first we choose ϕ^l_i based on the dynamic information of $\mathbf{v}^{r,\rho,l-1}$ and $\mathbf{r}^{l-1}$. We do this as indicated in the introduction, and as we explain in a more elaborative way now. Further details are provided below in the proof of the

main theorem. For each $1 \leq i \leq n$ and at each time step l the function ϕ^l_i is constructed as a sum $\phi^l_i = \phi^{v,l}_i + \phi^{r,l}_i$ where $\phi^{v,l}$ is designed in order to control the growth of $\mathbf{v}^{r,\rho,0,l}$ and $\mathbf{v}^{r,\rho,l}$ and $\mathbf{r}^l$ is designed in order to control the growth of $\mathbf{r}^l$. For all $1 \leq i \leq n$ the function $\phi^{v,l}_i$ is constructed such that $\phi^{v,l}_i$ gets the opposite sign of $v^{r,\rho,l-1}_i(l-1,.)$ whenever the modulus of the latter function exceeds a certain level. Assuming inductively that for $1 \leq i \leq n$ we have

$$\sup_{x\in\mathbb{R}^n} |v^{r,\rho,l-1}_i(l-1,.)| \leq C \tag{159}$$

for some $C > 0$ for all $1 \leq i \leq n$ we let $D^{v,l-1}_{+,i}$ be the set of coordinates where $v^{r,\rho,l-1}$ exceeds the level $C/2$, i.e., we have $D^{v,l-1}_{+,i} = \left\{x|v^{r,\rho,l-1}_i(l-1,x) \in \left[\frac{C}{2}, C\right]\right\}$ and we let $D^{v,l-1}_{-,i}$ be the set $\left\{x|v^{r,\rho,l-1}_i(l-1,x) \in \left[-C, -\frac{C}{2}\right]\right\}$, i.e., the set of arguments where $v^{r,\rho,l-1}$ is below the level $-C/2$. Then we define

$$\phi^{v,l}_i : (l-1,l] \times \mathbb{R}^n \to \mathbb{R}, \quad \phi^{v,l}_i(\tau,x) := \phi^{v,s,l}_i(x), \tag{160}$$

with a spatial function $\phi^{v,s,l}_i \in C^{1,2}_b$ using partions of unity such that

$$\phi^{v,s,l}_i(x) = \begin{cases} -1 \ \text{ if } x \in D^{v,l-1}_{i,+}, \\ \\ 1 \ \text{ if } x \in D^{v,l-1}_{i,-}, \end{cases} \tag{161}$$

and such that

$$\sup_{x\in\mathbb{R}^n} |\phi^{v,s,l}_i(x)| \leq 1. \tag{162}$$

Later, in the proof of theorem 2.4 we shall see that

$$|\phi^{v,s,l}_i|_{0,2} \leq C_s, \tag{163}$$

for some constant C_s independent of the time step number l. The construction of $\phi^{v,l}_i$ will ensure that the function $\mathbf{v}^{r,\rho,l} = \mathbf{v}^{r,\rho} + \mathbf{r}$ by the supremum of the initial data $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ on the time interval $[l-1,l]$. At the same time we have to ensure that the function $\mathbf{r}^l$ is bounded. Otherwise we cannot ensure that $\mathbf{v}^l := \mathbf{v}^{r,\rho,l} - \mathbf{r}^l$ is bounded. For the construction of $\phi^{r,l}_i$ we may assume from the previous time step that for $1 \leq i \leq n$ we have

$$\sup_{x\in\mathbb{R}^n} |r^{l-1}_i(l-1,.)| \leq C^0_r, \quad |r^{l-1}_i(l-1,.)|_{1,2} \leq C_r, \tag{164}$$

for some $C > 0$, and where C^0_r, C_r are constants independent of the time step number l which will be determined below. Recall that we defined sets $D^{r,l-1}_{+,i}$ and $D^{r,l-1}_{-,i}$ in order to define $\mathbf{r}^l$ such that its growth is controlled. The definition of $\phi^{r,l}_i$ involves subsets $D^{r,0,l-1}_{+,i} \subset D^{r,l-1}_{+,i}$ and $D^{r,0,l-1}_{+,i} \subset D^{r,l-1}_{-,i}$ sets such that $\left(D^{v,l-1}_{+,i} \cup D^{v,l-1}_{+,i}\right) \cap \left(D^{r,0,l-1}_{+,i} \cup D^{r,0,l-1}_{+,i}\right) = \oslash$. Defining $D^{v,l-1}_i =$

$D^{v,l-1}_{+,i}\cup D^{v,l-1}_{-,i}$ this latter relation is easily seen by rewriting these sets as subsets of

$$D^{r,0,l-1}_{+,i} := \left\lbrace x|r^l_i(l-1,x)\in \left[\frac{C}{2},C\right]~~x\not\in D^{v,l-1}_i\right\rbrace , \tag{165}$$

and

$$D^{r,0,l-1}_{-,i} := \left\lbrace x|r^l_i(l-1,x)\in \left[-C,-\frac{C}{2}\right]~~x\not\in D^{v,l-1}_i\right\rbrace . \tag{166}$$

Recall that we defined

$$\phi^{r,l}_i : (l-1,l]\times {\mathbb R}^n\rightarrow {\mathbb R},~\phi^{r,l}_i(\tau,x):=\phi^{r,s,l}_i(x), \tag{167}$$

with a spatial variable function $\phi^{r,s,l}_i\in C^{1,2}_b$ such that

$$\phi^{r,s,l}_i(x)=\left\lbrace \begin{array}{ll} -1~~\mbox{if}~x\in D^{r,0,l-1}_{i,+},\\ \\ 1~~\mbox{if}~x\in D^{r,0,l-1}_{i,-}.\end{array}\right. \tag{168}$$

We shall see in the proof of our theorem 2.4 that

$$\sup_{x\in {\mathbb R}^n}|\phi^{r,s,l}_i(x)|\leq 1,~~|\phi^{r,s,l}_i(x)|_{0,2}\leq C_s, \tag{169}$$

where the constant C_s is as above. Having constructed $\phi^l_i=\phi^{v,l}_i+\phi^{r,l}_i$ for all $1\leq i\leq n$ we solve the linear equation (157) for $\mathbf{r}^l$. Note hat all terms on the right side of (157) have the factor ρ_l except for ϕ^l_i. Choosing ρ_l small enough we can ensure that ϕ^l_i dominates the source terms in the critical regions of arguments where the moduli of the function r^{l-1}_i exceed a certain level. This will allow us to control the growth of the function $\mathbf{r}^l$ We shall see that classical a priori estimates ensure that

$$|r^l_i|_{1,2}\leq C_r \tag{170}$$

for some constant C_r independent of the time step number l. Then we plug in this function $\mathbf{r}^l$ into the right side of (154), and using (158) we get

$$\left\lbrace \begin{array}{ll} \frac{\partial v^{r,\rho,0,l}_i}{\partial \tau}-\rho_l\nu\sum_{j=1}^n \frac{\partial^2 v^{r,\rho,0,l}_i}{\partial x_j^2} +\rho_l\sum_{j=1}^n v^{r,\rho,l-1}_j(l-1,.)\frac{\partial v^{r,\rho,0,l}_i}{\partial x_j}=\\ \\ \psi^{l,0}_i=\phi^l_i+\left(\psi^{l,0}_i-\phi^l_i\right),\\ \\ v^{r,\rho,0,l}_i(l-1,.)=v^{r,\rho,l-1}_i(l-1,.).\end{array}\right. \tag{171}$$

Then we may represent the solution of (171) in terms of the fundamental solution Γ^l of the equation (155) and get

$$\begin{array}{ll} v^{r,\rho,0,l}_i(\tau,x)=\int_{{\mathbb R}^n}v^{r,\rho,l-1}_i(l-1,y)\Gamma^l(\tau,x;0,y)dy\\ \\ +\int_{l-1}^{\tau}\int_{{\mathbb R}^n}\left(\phi^l_i+\left(\psi^{l,0}_i-\phi^l_i\right) \right) (s,y)\Gamma^l(\tau,x;s,y)dsdy.\end{array} \tag{172}$$

The first term of this representation can be estimated using the maximum principle for the corresponding equation without source term. The second term can be estimated (and becomes small) if $\rho_l C_r$ becomes small (independently of the number l). Finally we have to ensure that the 'higher order corrections' of the functions $v_i^{r,\rho,0,l}$, $1 \leq i \leq n$, i.e., the functional series $\sum_{k=1}^{\infty} \delta \mathbf{v}^{r,\rho,k,l}$ which we have to add in order to compute the function $\mathbf{v}^{r,\rho,l}$ from its linear approximation $\mathbf{v}^{r,\rho,0,l}$, becomes small.

We prove

Theorem 2.4. *Given any dimension n and a viscosity constant $\nu > 0$ let $\mathbf{h}$ satisfy (147) for any $s \in \mathbb{R}$. Then there is a global classical solution*

$$\mathbf{v} \in \left[C^{1,2}\left([0,\infty) \times \mathbb{R}^n\right)\right]^n \tag{173}$$

to the Navier-Stokes equation system (4), (5), (6) which satisfies

$$v_i(t,.) \in H^2 \tag{174}$$

Remark 2.5. The proof below also shows that for all $t \geq 0$ and all $1 \leq i \leq n$.

$$|\partial_x^\alpha \mathbf{v}(t,.)| \leq \frac{C_{\alpha k}}{(1+|x|)^k} \text{ for } k \leq 5 \text{ and } |\alpha| \leq 2 \tag{175}$$

for some constants $C_{\alpha k} > 0$ which depend on dimension n the viscosity constant ν and the initial data $\mathbf{h}$. Moreover, below in Section 4 we shall use the proof of theorem 2.4 in order to improve the scheme above for numerical purpose and in order to adapt it to initial-boundary value problems. Especially, the function $\mathbf{r}$ is not needed for the algorithm since the the boundedness of the solution $\mathbf{v}$ is shown. The main purpose of the function $\mathbf{r}$ is to show boundedness of the solution $\mathbf{v}$. Once this is done it is not needed anymore and the algorithm simplifies.

Proof. We do the proof in four steps.

1) In a first step we prove existence of a bounded local classical solution $\mathbf{v}^{\rho,l}$ assuming existence of a regular function $\mathbf{v}^{\rho,l-1}(l-1,.)$, i.e., we prove existence of the local solution $\mathbf{v}^{\rho,l}$ on $[l-1,l] \times \mathbb{R}^n$ for $l-1 \geq 0$ where bounded data $v_i^{\rho,l-1}(l-1,.) \in C_b^2 \cap H^2$ are given. Here $C_b^2 \equiv C_b^2(\mathbb{R}^n)$ denotes the space of functions which have bounded derivatives up to second order. In this first step we set $\mathbf{r} = 0$. Recall that $\mathbf{v}^{\rho,l} = \mathbf{v}^{r,\rho,l} - \mathbf{r}$, and that $\mathbf{v}^{\rho,l} = \mathbf{v}^{0,\rho,l} = \mathbf{v}^{r,\rho,l}$ in case $\mathbf{r} = 0$. Moreover, we show that the local solution has certain polynomial decay at spatial infinity (as indicated in the statement of theorem 2.4 above) if the data have this property. Especially, we show that for each component function $v_i^{r,\rho,l}$ of the function $\mathbf{v}^{\rho,l}$ we have

$$v_i^{r,\rho,l}(\tau,.) \in H^2 \tag{176}$$

for all $\tau \in [l-1,l]$.

2) In a second step we show that the local iteration of the first step of this proof can be extended in order to show existence of a local solution $\mathbf{v}^{r,\rho,l}$ on $[l-1,l]\times\mathbb{R}^n$ for a certain class of functions $\mathbf{r}^l$ assuming existence of a regular function $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ and a bounded regular function $\mathbf{r}^l$ with a linearly bounded integral magnitude for r_i^l (as in (12)), i.e., we show existence of a bounded local solution $\mathbf{v}^{r,\rho,l}$ on $[l-1,l]\times\mathbb{R}^n$ for $l-1\geq 0$ where bounded data $v_i^{r,\rho,l-1}(l-1,.)\in C_b^2$ and $\mathbf{r}^{l-1}(l-1,.)\in C_b^2$ are given, and where

$$\int_{\mathbb{R}^n}\sum_{j,k=1}^{n}\Big|\left(\frac{\partial r_k^l}{\partial x_j}\frac{\partial r_j^l}{\partial x_k}\right)(t,y)\Big|dy<\infty. \tag{177}$$

Moreover, we show that the local solution has polynomial decay stated in theorem 2.4 if the initial data of the respective time step have this property.

3) In a third step we show that there exists a recursively defined bounded function $\mathbf{r}$ and a global iterative scheme of the solution $\mathbf{v}^{r,\rho}=\mathbf{v}+\mathbf{r}$. Especially, for each time step number l the bounded function $\mathbf{r}^l$ is defined via a linearized Navier-Stokes equation along with a source term $\phi^l=(\phi^l_1,\cdots,\phi^l_n)$ (among other source terms). These source terms ϕ^l_i are defined in terms of the functions $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ and $\mathbf{r}^{l-1}(l-1,.)$ which are known from the previous time step. They are defined in order to control the growth of the function $\mathbf{v}^{r,\rho}$. Here we use the fact that we have some freedom in order to define the functions r_i^l. Indeed these functions are defined in terms of the 'consumption terms' ϕ^l. We shall conclude that the functions $\mathbf{v}^{r,\rho,l}$ and $\mathbf{r}$ are globally bounded, i.e., there exists a constant $C>0$ independent of l such that for all $1\leq i\leq n$ and $l\geq 1$ we have

$$|v_i^{r,\rho,l}|_0\leq C. \tag{178}$$

Moreover at each time step $l\geq 1$ we shall ensure that

$$|r_i^l|_0\leq C,\ |v_i^{\rho,l}|_0=|v_i^{r,\rho,l}-r_i^l|_0\leq C. \tag{179}$$

The construction of the function $\mathbf{r}$ which equals $\mathbf{r}^l$ at each time step l is such that a) step 2 above can be applied with ρ_l and $\mathbf{r}^l$ as they are specified in this step, and b) such that the functions $\mathbf{r}^l$ and their spatial derivatives up to second order on $(l-1,l]\times\mathbb{R}^n$ are bounded and have a finite integral magnitude (177). The integral magnitude increases linearly in time. Hence the choice $\rho_l\sim\frac{1}{l}$ in order to have convergence of the local iteration scheme while the scheme is global.

4) In a fourth step we show the existence of a globally bounded classical solution of the Navier Stokes equation, i.e., we show that $\mathbf{v}^\rho=$

$\mathbf{v}^{r,\rho}+\mathbf{r}\in\left[C^{1,2}\left([0,\infty)\times\mathbb{R}\right)\right]^n$, and that for all $l\geq 1$ we have $\mathbf{v}^{\rho,l}=\mathbf{v}^{r,\rho,l}+\mathbf{r}^l\in\left[C^{1,2}_b\left([l-1,l]\times\mathbb{R}\right)\right]^n$ accordingly. This is not an immediate consequence of step 3) because for all $x\in\mathbb{R}^n$ the function $\tau\rightarrow\mathbf{r}(\tau,x)$ is only weakly differentiable at the integer values $l\in\mathbb{N}$ in general. Since the sequence (ρ_l) is chosen such that $\sum_{l=1}^N\rho_l\uparrow\infty$ as $N\uparrow\infty$ we conclude that the global solution $\mathbf{v}$ in original time coordinates exists.

2.1 step 1: proof of local existence of solutions at each time step

The proof of step i) is in two substeps. First we prove a contraction property on the domain $[l-1,l]\times\mathbf{R}^n$ for the series $(v_i^{\rho,k,l})_k,\leq i\leq n$ with

$$v_i^{\rho,k,l}=v_i^{\rho,0,l}+\sum_{m=1}^k\delta v_i^{\rho,m,l},\tag{180}$$

for a fixed time step l where we assume that $\mathbf{v}^{\rho,l-1}(l-1,.)\in C^{1,2}_b\cap H^2$ at the lth time step. In a first substep we construct an iteration in $C^{1,2}_b\left([l-1,l]\times\mathbb{R}^n\right)$ which is not an iteration in a Banach space. However, it is useful to have classical representations of approximating solutions in order to do some estimates. The limit function is Hölder continuous if we have have a certain decay at infinity. Therefore, in a second substep of this first step we show that we have a certain decay at spatial infinity such that we can use one of the method a) (in case $n=3$ with Hilbert space theory, i.e., showing convergence of the series $v_i^{r,\rho,k,l}$ in the Hilbert space H^2, or showing convergence of the series $v_i^{r,\rho,k,l}$ in with respect to $H^{s,p}$ Banach spaces in the general case) or the even more elementary method b) (where n is an arbitrary dimension n) in order to prove the existence of a local fixed point. The method c) of the introduction is considered in a subsequent paper- it becomes important if one considers the Navier-Stokes equation on manifolds.

Remark 2.6. Note that the argument of this first step is local. We establish some property for the function $\mathbf{v}^{\rho,l}$ assuming that some property holds for the function $\mathbf{v}^{\rho,l-1}(l-1,.)$ of the previous time step which serves as initial data for the lth time step, i.e., $\mathbf{v}^{\rho,l}(l-1,.)=\mathbf{v}^{\rho,l-1}(l-1,.)$. This means that the argument here is about local existence at an arbitrary time step l. The global existence (induction over l) is then considered with the introduction of the function $\mathbf{r}$ in the third step of this proof.

For $l\geq 1$ assume that $\mathbf{v}^{\rho,l-1}(l-1,.)$ has been computed, i.e., the initial data of equation (81) have been computed, and assume that for all $1\leq i\leq n$

$$|v_i^{\rho,l-1}|_0\leq C_0^{l-1},\ |v_i^{\rho,l-1}|_{0,1}\leq C_1^{l-1}|,\ |v_i^{\rho,l-1}|_{0,2}\leq C_{0,2}^{l-1},\ |v_i^{\rho,l-1}|_{1,2}\leq C_{1,2}^{l-1}\tag{181}$$

for some constant $C_0^{l-1},\ C_1^{l-1},\ C_{0,2}^{l-1},\ C_{1,2}^{l-1}$ which are given from the previous time step $l-1$. In order to get estimates on Hilbert spaces H^2 we use an inductive assumption for the H^2-norm, i.e. we assume that

$$\sum_{i=1}^{n}\sum_{|\alpha|\leq 2}\int_{\mathbb{R}^n}|v^{\rho,l-1}_{i,\alpha}(l-1,x)|^2dx\leq C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}. \tag{182}$$

The reason for this bound -which is linear in time- will become apparent in step 3 of this proof.

Furthermore, in order to apply the alternative method b) described in the introduction we assume that initial data function of the previous time step $x\rightarrow \mathbf{v}^{\rho,l-1}(l-1,x)=\left(v^{\rho,l-1}_1(l-1,x),\cdots,v^{\rho,l-1}_n(l-1,x)\right)^T$ with components $v^{\rho,l-1}_i$ which for all $1\leq i\leq n$ satisfy

$$|\partial^{\alpha}_x\mathbf{v}^{\rho,l-1}(l-1,x)|\leq \frac{C_{\alpha k}}{(1+|x|)^k} \tag{183}$$

for all $x\in\mathbb{R}^n$, and for all multiindices $|\alpha|\leq 2$ and and integers $1\leq k\leq 5$ with some constants $C_{\alpha k}$.

Remark 2.7. If there is an external force term $\mathbf{f}_{ex}$ included on the right then we may assume that

$$|\partial^m_t\partial^{\alpha}_x\mathbf{f}_{ex}(t,x)|\leq \frac{C_{\alpha mk}}{(1+|x|+t)^k}\qquad\text{for all}\qquad (t,x)\in[0,\infty)\times\mathbb{R}^n, \tag{184}$$

for any multiindices α, integers k, and nonnegative integers m, and with some constants $C_{\alpha mk}$.

In case $l=1$ (the first time step) we have $\mathbf{v}^{\rho,l-1}(l-1,.)=\mathbf{h}$ and the constants $C_0^0,\ C_1^0,C_{0,2}^0$ are given in terms of upper bounds of the components of the functions h_i and derivatives of the functions h_i. The series (180) can be generated by an iterative application of the map

$$F_l:\mathbf{f}\rightarrow\mathbf{v}^{f,\rho,l}, \tag{185}$$

starting with the value of F_l applied to the initial data of step l, i.e., starting with the function $\mathbf{v}^{\rho,0,l}=F_l\left(\mathbf{v}^{\rho,l-1}(l-1,.)\right)$. Here, $\mathbf{v}^{\rho,l-1}(l-1,.)$ represent the final data of the previous step or the data $\mathbf{h}$ in case $l=1$. The function $\mathbf{v}^{f,\rho,l}$ is determined by the solution of the equation

$$\left\lbrace\begin{array}{ll}
\frac{\partial v^{f,\rho,l}_i}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v_i^{f,\rho,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n f_j\frac{\partial v_i^{f,\rho,l}}{\partial x_j}=\\
\\
\qquad\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial f_k}{\partial x_j}\frac{\partial f_j}{\partial x_k}\right)(\tau,y)dy,\\
\\
\mathbf{v}^{f,\rho,l}(l-1,.)=\mathbf{v}^{\rho,l-1}(l-1,.).
\end{array}\right. \tag{186}$$

We want to show that the map F_l is a contraction on this series with respect to some appropriate norm, i.e., we want to show that

$$|\delta \mathbf{v}^{\rho,k,l}| \leq \frac{1}{4}|\delta \mathbf{v}^{\rho,k-1,l}| \tag{187}$$

for some suitable norms $|.|$ and $k \geq 1$, and where we start the series with $\mathbf{v}^{\rho,0,l} = F_l\left(\mathbf{v}^{\rho,l-1}\right)$. The norms are the $|.|_{1,2}$ norm in order to have classical representations of approximating solutions in terms of certain fundamental solutions, and the $|.|_{H^2}$- and $|.|_{H^{2,p}}$-norms in order finish the estimate for the limit according to method a). Furthermore, we shall show that we have some decay at infinity. We shall also see that we have a linear growth of the integral magnitude for $\mathbf{v}^{\rho,l}$, i.e., we have a linear growth upper bound with respect to time of the magnitude

$$\int_{\mathbb{R}^n} \sum_{j,k=1}^{n} \Big| \left(\frac{\partial v_k^{\rho,l}}{\partial x_j} \frac{\partial v_j^{\rho,l}}{\partial x_k} \right) \Big|(t,y)dy. \tag{188}$$

Since products of functions may be pointwise estimated by half the sum of the squares of the factor functions, the linear growth upper bound of (188) follows from the linear growth upper bound of the H^2-norm of the solution function. Note that the term (188) is related to the integral term of the Leray projction form of the Navier-Stokes equation naturally. Since (188) can be estimated quite naturally in term of H^2 norms, it is quite natural to estimate the solution of the Navier-Stokes equation in the H^2 norm. Interestingly, in case of $n = 3$ this is also sufficient in order to have an embedding in Hölder continuous spaces. Recall that in case $l = 1$ we have $\mathbf{v}^{\rho,l-1} = \mathbf{h}$. In order to prove a contraction property of type (187) we consider the map $F_l(\mathbf{f}) = \mathbf{v}^{f,\rho,l}$ for general $\mathbf{f} \in \left[C_b^{1,2}\right]^n$ first, where $\mathbf{v}^{f,\rho,l}$ satisfies the equation (186). We assume that the data $v_i^{\rho,l-1}(l-1,.) \in C_b^{1,2}$ are bounded (this is certainly true in the case $l = 1$, where $\mathbf{v}_i^{\rho,l-1}(l-1,.) = \mathbf{h}$). Note again that in this first step we let $\mathbf{r} = 0$. We consider the methods a) and b) of the introduction and consider first the normed space $C_b^{1,2}\left(D_l^\tau\right)$ along with the norm $|.|_{1,2}$ of the introduction. Note that this normed space, i.e., $C_b^{1,2}$ equipped with the norm $|.|_{1,2}$ is not a Banach space, but we can proceed according to the methods a) or b) of the introduction. We shall determine $\mathbf{v}^{\rho,l} = \left(v_1^{\rho,l}, \cdots, v_n^{\rho,l}\right)^T$ as a limit of a functional series with members in $C_b^{1,2}$, and with an additional decay at infinity, i.e., for all $1 \leq i \leq n$ we construct $v_i^{\rho,l}$ as a limit of the approximations $v_i^{\rho,k,l}$

$$C_b^{1,2}\left(D_l^\tau\right) \ni v_i^{\rho,k,l} = v_i^{\rho,0,l} + \sum_{m=1}^{k} \delta v_i^{\rho,m,l}, \tag{189}$$

where we prove in a second substep that all functions $v_i^{\rho,k,l}$ have a certain decay at infinity. Here, the function $\mathbf{v}^{\rho,0,l} = \left(v_1^{\rho,0,l}, \cdots, v_n^{\rho,0,l}\right)^T$ solves

$$\begin{cases} \frac{\partial v_i^{\rho,0,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{\rho,0,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{\rho,l-1} \frac{\partial v_i^{\rho,0,l}}{\partial x_j} \\ \\ = \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{\rho,l-1}}{\partial x_j} \frac{\partial v_j^{\rho,l-1}}{\partial x_k}\right)(\tau,y)dy, \\ \\ \mathbf{v}^{\rho,0,l}(l-1,.) = \mathbf{v}^{\rho,l-1}(l-1,.). \end{cases} \tag{190}$$

Furthermore, for $k \geq 1$ we consider the difference functions $\delta \mathbf{v}^{\rho,k,l} = \mathbf{v}^{\rho,k,l} - \mathbf{v}^{\rho,k-1,l}$ where for all k we have $\mathbf{v}^{\rho,k,l} = \mathbf{v}^{r,\rho,k,l}|_{\mathbf{r}=0}$. Note that we have $\delta \mathbf{v}^{\rho,k,l}$ recursively defined to be solutions of the equations

$$\begin{cases} \frac{\partial \delta v_i^{\rho,k,l}}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \delta v_i^{\rho,k,l}}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{\rho,k-1,l} \frac{\partial \delta v_i^{\rho,k,l}}{\partial x_j} = \\ \\ -\rho_l \sum_{j=1}^n \delta v_j^{\rho,k-1,l} \frac{\partial v_i^{\rho,k,l}}{\partial x_j} \\ \\ +\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{\rho,k-1,l}}{\partial x_j} \frac{\partial v_j^{\rho,k-1,l}}{\partial x_k}\right)(\tau,y)dy, \\ \\ -\rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial v_k^{\rho,k-2,l}}{\partial x_j} \frac{\partial v_j^{\rho,k-2,l}}{\partial x_k}\right)(\tau,y)dy, \\ \\ \delta \mathbf{v}^{\rho,k,l}(l-1,.) = 0. \end{cases} \tag{191}$$

Here, for $k=1$ we denote $v_i^{\rho,k-2,l} = v_i^{\rho,-1,l} := v_i^{\rho,l-1}$. Solving these equations recursively leads us successively to the functions

$$\mathbf{v}^{\rho,k,l} = \mathbf{v}^{\rho,0,l} + \sum_{m=1}^k \mathbf{v}^{\rho,m,l} \tag{192}$$

which approximate the local solution $\mathbf{v}^{\rho,l}$. In order to apply the methods a) and b) of the introduction we show first that

$$|\delta \mathbf{v}^{\rho,k,l}|_{1,2} \leq \frac{1}{4} |\delta \mathbf{v}^{\rho,k-1,l}|_{1,2}. \tag{193}$$

Then in order to apply method a) of the introduction we show that for all $\tau \in [l-1,l]$ we have

$$|\delta \mathbf{v}^{\rho,k,l}(\tau,.)|_{H^2} \leq \frac{1}{4} |\delta \mathbf{v}^{\rho,k-1,l}(\tau,.)|_{H^2}. \tag{194}$$

uniformly in τ. We shall also consider how polynomial decay at infinity is preserved by the local scheme if the initial data from the previous time step have this property. In this work on the Navier-Stokes equation system we use methods a) and b) of the introduction in order to show the existence of time-local solutions for general dimension by elementary means. Recall that in case of method b) it is shown that the members of the functional series $(\mathbf{v}^{\rho,k,l})_k$ satisfy a specific polynomial decay at infinity such that a spatial transformation leads us to a functional series in a Banach space, and this method applies for any dimension n. Hence we have two different proofs in the case $n=3$ corresponding to the methods a) and b) of the introduction.

Remark 2.8. Locally a sharper estimate with respect to the Hölder space $C_{1+\alpha/2,2+\alpha}\left(D_l^\tau\right)$ is possible. However, it is easier to control the growth with respect to the $|.|_{1,2}$ norm. This control of the growth is essential for the global existence proof. Classical Schauder estimates introduce a potential term with a certain sign, and this would lead to a exponentially decreasing series of time step size numbers ρ_l if applied naively. In this case the sum of the time step size numbers ρ_l is finite and the scheme is not global in time. Hence, method c) of the introduction may be applied locally, but a globally some additional work is needed.

Note that in case $n=3$ it is essential to show in the second substep that $v_i^{\rho,k,l}(\tau,.)\in H^2\left(\mathbb{R}^n\right)$ for all $1\leq i\leq n$, because the limit $v_i^{\rho,l}(\tau,.)\in H^2\left(\mathbb{R}^n\right)$ for all $1\leq i\leq n$ (uniformly in τ) implies that

$$v^{\rho,l}(\tau,.)_i\in C^\alpha\subset H^2\left(\mathbb{R}^3\right) \tag{195}$$

for some $\alpha>0$ uniformly in τ for all $1\leq i\leq n$.

Remark 2.9. In order to extend the proof to general dimension n we may use the estimate For all $s\in\mathbb{R}^n$ and $p\in(1,\infty)$

$$H^{s,p}\left(\mathbb{R}^n\right)\subset C_*^r\left(\mathbb{R}^n\right) \tag{196}$$

for $r=s-\frac{n}{p}$. Hilbert space estimates for $n>3$ are harder to obtain. The relation $v_i^{\rho,k,l}(\tau,.)\in H^s\left(\mathbb{R}^n\right)$ for some $s\geq m+\frac{n}{2}+\alpha$ and some $m\geq 0$, and some $\alpha\in(0,1)$ for all $1\leq i\leq n$ implies

$$v_i^{\rho,l}(\tau,.)\in C^\alpha\subset H^s\left(\mathbb{R}^n\right). \tag{197}$$

Concerning method b) and c) of the introduction we note that the spatial transformation on a finite domain does not preserve the uniform ellipticity of the operator (represented by the positive constant ν in the original equation). In this respect the method a) is a little bit more elementary than methods b) and c) of the introduction, because we do not need a spatial coordinate transformation in this case. In order to check the contraction property of the functional series $\left(\mathbf{v}^{\rho,k,l}\right)_{k\geq 0}$ we check the contraction property of the

functional (185) on an appropriate functional subset of the function space $\left[C^{1,2}_b\left(\mathbb{R}^n\right)\right]^n$. First let us stick with the original spatial coordinates and consider the equation for $v^{f,\rho,l}_i$ more closely. Using relation (108) from equation (106) we get

$$
\begin{array}{ll}
v^{f,\rho,l}_i(\tau,x)-v^{g,\rho,l}_i(\tau,x)= & \\
=-\rho_l\int_{(l-1)}^{\tau}\int_{\mathbb{R}^n}\sum_{j=1}^n\left(f_j-g_j\right)(s,y)\frac{\partial v^{g,\rho,l}_i}{\partial x_j}(s,y)\Gamma^l_f(\tau,x;s,y)dyds+ & \\
\int_{l-1}^{\tau}\rho_l\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K_{n,i}(z-y)\times & (198)\\
\left(\left(\sum_{j,k=1}^n\left(f_{k,j}+g_{k,j}\right)(s,y)\right)\times\right. & \\
\left.\left(\sum_{j,k=1}^n\left(f_{j,k}-g_{j,k}\right)(s,y)\right)\right)\Gamma^l_f(\tau,x;s,z)dydzds, &
\end{array}
$$

where Γ^l_f is the fundamental solution of the equation

$$
\frac{\partial\Gamma^l_f}{\partial\tau}-\rho_l\nu\Delta\Gamma^l_f+\rho_l\sum_{j=1}^n f_j\frac{\partial\Gamma^l_f}{\partial x_j}=0 \tag{199}
$$

on the domain D^{τ}_l.

Here, and in the following we write $\left(f_j-g_j\right)(s,y):=f_j(s,y)-g_j(s,y)$, $\left(f_{j,k}-g_{j,k}\right)(s,y)=f_{j,k}(s,y)-g_{j,k}(s,y)$ etc. for simplicity of notation. We show that ρ_l can be chosen such that the iterations

$$
F^m_l(\mathbf{v}^{\rho,l-1})\in S_0, \tag{200}
$$

where S_0 is a subset of $C^{1,2}_b$, i.e.,

$$
\mathbf{f},\mathbf{g}\in\left\{\mathbf{f}||f_i|_{1,2}\leq 2C^{l-1}_{1,2}\&|f_i(\tau,.)|_{H^2}\leq 2C^{l-1}\text{ unif. }\forall\tau\in[l-1,l]\right\}=:S_0, \tag{201}
$$

along with

$$
|v^{\rho,l-1}_i|_{1,2}\leq C^{l-1}_{1,2}. \tag{202}
$$

The choice of the space S_0 is related to the fact that we use classical representations of approximating solutions which may be estimated in terms of Gaussian estimates. These Gaussian estimates are convolutions which are estimated in terms applying Young's in equality (cf. below).

Remark 2.10. You may ask why we consider the contraction in the space $C^{1,2}_b$ with norm $|.|_{1,2}$ at all, since this is not a Banach space. The reason is that classical representations are very useful when we establish a local scheme. This will become apparent in step 3 of this proof.

The factor 2 in the definition of S_0 is due to the fact that the first order coefficients vary with the the local iteration number k (see below). The requirement that $h_i \in H^2$ in (201) is due to the fact that we need to establish a contraction with respect to the H^2 norm in order to apply method a) of the introduction. In order to get a uniform bound of this first order coefficients which are independent of the iteration number k we establish an estimate of the form

$$\sum_{i=1}^{n} |v_i^{f,\rho,l} - v_i^{g,\rho,l}|_{1,2} \leq \frac{1}{4} |\mathbf{f} - \mathbf{g}|_{1,2}^{n}. \tag{203}$$

Recall from above that we equip the vector valued functions of the classical function space $C_b^{1,2}$ with the norm

$$|\mathbf{f}|_{1,2}^{n} = \sum_{i=1}^{n} |f_i|_{1,2} \tag{204}$$

(we may also consider the maximum etc. instead of the sum, of course, but such considerations do matter only if we consider implementations of associated algorithms; the reference to the domain D_l^τ is not denoted if it is clear from the context for sake of simplicity of notation).

Remark 2.11. Note that the estimate (203) is one step of our construction of a solution in order to apply the methods of item a) and b) of the introduction. Locally in time there is a refined estimate for Hölder spaces, i.e., we can get estimates of the form

$$\sum_{i=1}^{n} |v_i^{f,\rho,l} - v_i^{g,\rho,l}|_{1+\alpha/2,2+\alpha} \leq c |\mathbf{f} - \mathbf{g}|_{1+\alpha/2,2+\alpha}^{n}. \tag{205}$$

Here, the Hölder norm is

$$|\mathbf{f}|_{1+\alpha/2,2+\alpha}^{n} = \sum_{i=1}^{n} |f_i|_{1+\alpha/2,2+\alpha}. \tag{206}$$

Then the method c) of the introduction can be used in order to construct a solution at each time step l. The application of the methods a) and b) is described below.

In order to prove the contraction property (203) it is essential to estimate the second order derivatives of the difference $v_i^{f,\rho,l} - v_i^{g,\rho,l}$, i.e., it is essential to estimate

$$v_{i,k,m}^{f,\rho,l} - v_{i,k,m}^{g,\rho,l} = \frac{\partial^2}{\partial x_k \partial x_m} \left(v_i^{f,\rho,l} - v_i^{g,\rho,l} \right). \tag{207}$$

Note that we consider the case $r_i^l = 0$ for all $1 \leq i \leq n$ in this first step of the proof.

We consider the two summands on the right side of (198) separately. Accordingly, let us define

$$\begin{array}{ll} v_i^{f,\rho,l,1}(\tau,x) - v_i^{g,\rho,l,1}(\tau,x) := \\ = -\rho_l \int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \sum_{j=1}^n \left(f_j - g_j\right)(s,y) \frac{\partial v_i^{g,\rho,l}}{\partial x_j}(s,y) \Gamma_f^l(\tau,x;s,y) dy ds, \end{array} \tag{208}$$

and

$$\begin{array}{ll} v_i^{f,\rho,l,2}(\tau,x) - v_i^{g,\rho,l,2}(\tau,x) := \\ = \int_{(l-1)}^{\tau} \rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{n,i}(z-y) \times \\ \left(\left(\sum_{j,k=1}^n \left(f_{k,j}(s,y) + g_{k,j}(s,y)\right)\right) \times \right. \\ \left. \left(\sum_{j,k=1}^n \left(f_{j,k}(s,y) - g_{j,k}(s,y)\right)\right) \right) \Gamma_f^l(\tau,x;s,z) dy dz ds, \end{array} \tag{209}$$

Since $\mathbf{f},\mathbf{g} \in \left[C_b^{1,2}\left(D_l^{\tau}\right)\right]^n$ we know that the classical fundamental solution Γ_f^l exists in its classical Levy expansion form. Since the initial data are $C_b^{1,2}$ we know that v_i^g is in $C_b^{1,2}\left(D_l^{\tau}\right)$ for all $1 \leq i \leq n$.

Especially, $(\tau,x) \rightarrow \sum_{j=1}^n \left(f_j - g_j\right)(\tau,x) v_{i,j}^{g,\rho,l}(\tau,x)$ is Hölder continuous in the spatial variables uniformly in $\tau \in [l-1,l]$.

Hence, classical Levy expansion theory of fundamental solutions (cf. [3]) shows that the function $v_i^{f,\rho,l,1}(\tau,x) - v_i^{g,\rho,l,1}(\tau,x)$ has spatial derivatives up to second order and that for spatial derivative of second order of the first summand we have the representation

$$\begin{array}{ll} v_{i,j,k}^{f,\rho,l,1}(\tau,x) - v_{i,j,k}^{g,\rho,l,1}(\tau,x) := \\ -\rho_l \int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \sum_{j=1}^n \left(f_j - g_j\right)(s,y) \frac{\partial v_i^{g,\rho,l}}{\partial x_j}(s,y) \Gamma_{f,j,k}^l(\tau,x;s,z) dy dz ds. \end{array} \tag{210}$$

We shall see below how to rewrite this with only one spatial derivative of the adjoint of the fundamental solution Γ_f^l. Next concerning the function $v_i^{f,\rho,l,2} - v_i^{g,\rho,l,2}$ for fixed time parameter s the integral function part

$$\begin{array}{l} x \rightarrow \int_{\mathbb{R}^n} K_{n,i}(x-y) \times \\ \left(\left(\sum_{j,k=1}^n \left(f_{k,j}(s,y) + g_{k,j}(s,y)\right)\right) \times \right. \\ \left. \left(\sum_{j,k=1}^n \left(f_{j,k}(s,y) - g_{j,k}(s,y)\right)\right) \right) dy \end{array} \tag{211}$$

looks formally like the solution of a Poisson equation

$$- \Delta p = \sum_{j,k=1}^{n} \left(f_{k,j} f_{j,k} - g_{k,j} g_{j,k} \right). \tag{212}$$

It is indeed a solution if the right side is in L^1, if pointwise products of functions of type $x \to f_k(s,x)$ and $x \to g_k(s,x)$ are in $H^{1,1}$. Again, since

$$|f_{k,j} f_{j,k}|(\tau,x) \leq \frac{1}{2} \left(|f_{k,j}|^2(\tau,x) + |f_{j,k}|^2(\tau,x) \right) \tag{213}$$

it is sufficient that we have $f_k(\tau,.), g_k(\tau,.) \in H^2$ (even $f_k(\tau,.), g_k(\tau,.) \in H^1$ is sufficient at this stage). We shall see that the functions which are produced by iterations of the functional F_l starting with $v_i^{\rho,l}$ satisfy this requirement. Therefore in the following we may assume that this requirement is satisfied such that the function in (211) exists. We shall consider this in more detail later in this first step of the proof, and see that it follows from our polynomial decay condition on the initial data $v_i^{\rho,l-1}$ and the properties of the functional F_l. The following fact is rather classical.

Lemma 2.12. *Let $n \geq 3$, and let $f \in L^1(\mathbb{R}^n)$. Let K be the fundamental solution for the Laplacian. Then*

$$(t,x) \to u(t,x) := \int_{\mathbb{R}^n} f(y) K(x-y) dy \tag{214}$$

is a well-defined locally integrable function which solves the Poisson equation on $\mathbb{R}^n$ with data f, i.e.,

$$\Delta u = f. \tag{215}$$

Remark 2.13. If $n = 2$ then we need the stronger assumption $\int f(z) \log |z| dz < \infty$ and the theorem is still true.

Hence, if $f_j, g_j \in C_b^{1,2}$, then the function (211) is indeed the solution of (212). Concerning the regularity of the solution u of the Poisson equation we have another classical result due to Hölder himself.

Lemma 2.14. *Let n, k be some positive integers. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $f \in C^{k+\alpha}(\Omega)$. If u is a distributive solution of*

$$\Delta u = f, \tag{216}$$

then $u \in C^{2+\alpha+k}(\Omega)$.

If $f_j, g_k \in C_b^{1,2}$ for $1 \leq j, k \leq n$, then the first order derivatives $f_{j,l}, g_{k,m}$ are Hölder continuous. This implies that the function of equation (211) is

in $C^{1+\alpha}$ (note the derivative of the kernel K). Hence, the integral

$$\begin{aligned}
&v^{f,\rho,l,2}_{i,j,k}(\tau,x)-v^{g,\rho,l,2}_{i,j,k}(\tau,x):=\\
&=\int_{(l-1)}^{\tau}\rho_l\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K_{n,i}(z-y)\times\\
&\left(\left(\sum_{j,k=1}^n\left(f_{k,j}(s,y)+g_{k,j}(s,y)\right)\right)\times\right.\\
&\left.\left(\sum_{j,k=1}^n\left(f_{j,k}(s,y)-g_{j,k}(s,y)\right)\right)\right)\Gamma^l_{f,j,k}(\tau,x;s,z)dydzds
\end{aligned}\tag{217}$$

exists and represents the second order derivatives of the second term (209). The equations (210) and (217) contain second order derivatives of the fundamental solution Γ^l_f. Note that the function

$$\begin{aligned}
&z\rightarrow\int_{\mathbb{R}^n}K_{n,i}(z-y)\times\\
&\left(\left(\sum_{j,k=1}^n\left(f_{k,j}+g_{k,j}\right)(s,y)\right)\times\right.\\
&\left.\left(\sum_{j,k=1}^n\left(f_{j,k}-g_{j,k}\right)(s,y)\right)\right)dy
\end{aligned}\tag{218}$$

is Hölder continuous, hence the function $v^{f,\rho,l,2}_{i,j,k}-v^{g,\rho,l,2}_{i,j,k}$ is continuous. In order to do the contraction estimate it is useful to rewrite equation (217) and equation (210).

Note that the function (218) is solution of a Poisson equation with a right side which is Hölder uniformly in τ. Hence, since the function (218) is in $C^{1+\alpha}$ with respect to the spatial variables and uniformly in τ, we may rewrite (217) shifting differentiation from the fundamental solution to the function (218) (shifting one differentiation is enough). This way we have only first order derivatives of the fundamental solution in the representation of $v^{f,\rho,l,2}_{i,j,k}-v^{g,\rho,l,2}_{i,j,k}$, and first order derivatives of the fundamental solution are integrable and can be handled more easily. Note that the derivative of the fundamental solution is with respect to the x-variables and the integrals in (210) and (217) are with respect to the conjugate y variables. In order to shift the differentiation one may use a relation of the fundamental solution and its adjoint, i.e., the relation

$$\Gamma^l_f(\tau,x;s,y)=\Gamma^{l,*}_f(s,y;\tau,x).\tag{219}$$

where $\Gamma^{l,*}_f$ is the adjoint of Γ^l_f. Application of derivatives of this relation (219) is exactly what we need. However, let us show this in detail in terms of the Levy expansion of the fundamental solution (as a by-product, this is also an independent proof of (219)). First we shift the derivative $K_{,i}$ of the

kernel K in (217). This is no problem since the inner integral with respect to z represents a convolution. Hence, we have

$$\begin{array}{l} v^{f,\rho,l,2}_{i,m,p}(\tau,x)-v^{g,\rho,l,2}_{i,m,p}(\tau,x):=\\ =-\int_{(l-1)}^{\tau}\rho_l\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K_n(z-y)\times\\ \left(2\left(\sum_{j,k=1}^n\left(f_{k,j,i}+g_{k,j,i}\right)(s,y)\right)\times\right.\\ \left.\left(\sum_{j,k=1}^n\left(f_{j,k}-g_{j,k}\right)(s,y)\right)\right)\Gamma^l_{f,m,p}(\tau,x;s,z)dydzds. \end{array} \tag{220}$$

Remark 2.15. Note that in this representation we have even only first derivatives of the difference $(f_{j,k}(s,y)-g_{j,k}(s,y))$. This may be convenient in some circumstances (if we want a sharper estimate) but it is not necessary in case of our modest goal, i.e. the contraction on a subset of $C^{1,2}_b$ with respect to $|.|_{1,2}$ norm.

Now let us show explicitly how we shift one derivative of $\Gamma^l_{f,m,p}$ First, recall from classical theory that the fundamental solution Γ^l_f in the Levy expansion form is given by

$$\Gamma^l_f(\tau,x;s,y):=N^l(\tau,x;s,y)+\int_s^{\tau}\int_{\mathbb{R}^n}N^l(\tau,x;\sigma,\xi)\phi(\sigma,\xi;s,y)d\sigma d\xi, \tag{221}$$

where

$$N^l(\tau,x;s,y)=\frac{1}{\sqrt{4\pi\rho_l\nu(\tau-s)}^n}\exp\left(-\frac{|x-y|^2}{4\rho_l\nu(\tau-s)}\right), \tag{222}$$

and ϕ is a recursively defined function which is Hölder continuous in x, i.e.,

$$\phi(\tau,x;s,y)=\sum_{m=1}^{\infty}(L_lN^l)_m(\tau,x;s,y), \tag{223}$$

along with the recursion

$$\begin{array}{l} (L_lN^l)_1(\tau,x;s,y)=L_lN^l(\tau,x;s,y)\\ =\frac{\partial N^l}{\partial\tau}-\rho_l\nu\Delta N^l+\rho_l\sum_{j=1}^n f_j\frac{\partial N^l}{\partial x_j}\\ =\rho_l\sum_{j=1}^n f_j\frac{\partial N^l}{\partial x_j},\\ (LN^l)_{m+1}(\tau,x):=\int_s^t\int_{\Omega}\left(LN^l(\tau,x;\sigma,\xi)\right)_m LN^l(\sigma,\xi;s,y)d\sigma d\xi. \end{array} \tag{224}$$

First we look at (220). We may write

$$\begin{array}{l} v^{f,\rho,l,2}_{i,m,p}(\tau,x)-v^{g,\rho,l,2}_{i,m,p}(\tau,x):=\\ =-\int_0^{\tau}\rho_l\int_{\mathbb{R}^n}S(s,y)\Gamma^l_{f,m,p}(\tau,x;s,z)dydzds \end{array} \tag{225}$$

along with a Hölder continuous function S

$$S(s,y):=\int_{\mathbb{R}^n}K_n(z-y)\times$$
$$\left(2\left(\sum_{j,k=1}^n\left(f_{k,j,i}+g_{k,j,i}\right)(s,y)\right)\times\right.\tag{226}$$
$$\left.\left(\sum_{j,k=1}^n\left(f_{j,k}-g_{j,k}\right)(s,y)\right)\right)dz,$$

which is uniformly Hölder continuous with respect to the time parameter s. Now we may write

$$v^{f,\rho,l,2}_{i,m,p}(\tau,x)-v^{g,\rho,l,2}_{i,m,p}(\tau,x):=$$
$$=-\int_0^\tau\rho_l\int_{\mathbb{R}^n}S(s,y)N^l_{,m,p}(\tau,x;s,y)dsdy$$
$$+\int_s^\tau\int_{\mathbb{R}^n}S(s,y)N^l_{,m,p}(\tau,x;\sigma,\xi)\left(\sum_{q=1}^\infty(LN^l)_q(\sigma,\xi;s,y)\right)S(s,y)d\sigma d\xi dydzds.\tag{227}$$

We may do the partial integral term by term and get exactly the (second order differential of the) Levy expansion for the adjoint. Well it does not matter so much that we get exactly the adjoint. What matters is that we get an absolutely convergent sum where only first order derivatives of the kernel N_l are involved, since such first order derivatives are locally integrable and can be estimated easily. We may denote the result of this partial integration for the Levy expansion sum by $\Gamma^{l,*}_f$ (which happens to be the adjoint). Hence, from equation (210) we get

$$v^{f,\rho,l,1}_{i,k,m}(\tau,x)-v^{g,\rho,l,1}_{i,k,m}(\tau,x)=$$
$$=-\rho_l\int_0^\tau\int_{\mathbb{R}^n}\sum_{j=1}^n\left(f_j-g_j\right)(s,y)v^{g,\rho,l}_{i,j}(s,y)\Gamma^l_{f,k,m}(\tau,x;s,y)dyds$$
$$=\rho_l\int_{(l-1)}^\tau\int_{\mathbb{R}^n}\sum_{j=1}^n\left(f_j-g_j\right)(s,y)v^{g,\rho,l}_{i,j,m}(s,y)\Gamma^{l,*}_{f,k}(s,y;\tau,x)dyds$$
$$+\rho_l\int_{(l-1)}^\tau\int_{\mathbb{R}^n}\sum_{j=1}^n\left(f_{j,m}-g_{j,m}\right)(s,y)v^{g,\rho,l}_{i,j}(s,y)\Gamma^{l,*}_{f,k}(s,y;\tau,x)dyds.\tag{228}$$

Next, we estimate the integral with integrand $\Gamma^{l,*}_{f,k}$ by a constant C_{Γ_f} using estimates for local integrability of the first order derivatives of the fundamental solution in Levy expansion form. Here, we may use standard estimates of the form

$$|\Gamma^{l,*}_{f,k}|\leq\frac{C_f}{\sqrt{4\pi(\tau-s)}^{n+1}}\exp\left(-\lambda_f\frac{|x-y|^2}{4\rho_l\nu(\tau-s)}\right),\tag{229}$$

for some constants C,λ and for $|x-y|\geq 1$ in order to have integrability at

infinity, and then use

$$|N^l(\tau - s, x, y)| \leq \frac{C}{(\tau - s)^{\alpha}|x - y|^{n+1-2\alpha}} \tag{230}$$

for $\alpha \in (0.5, 1)$ on a domain where $|x - y| \leq 1$ in order to have local integrability. We can formulate have both standard estimates independently of $\mathbf{f} \in S_0$. We get

$$\sup_{\mathbf{f} \in S_0} |\Gamma^{l,*}_{f,k}| \leq \frac{C}{\sqrt{4\pi(\tau - s)}^{n+1}} \exp\left(-\lambda \frac{|x - y|^2}{4\rho_l \nu(\tau - s)}\right), \tag{231}$$

where

$$C = \sup_{\mathbf{f} \in S_0} C_f, \tag{232}$$

and

$$\lambda = \inf_{\mathbf{f} \in S_0} \lambda_f, \tag{233}$$

in order to Furthermore, we get

$$|\Gamma^{l,*}_{f,k}(\tau, x; s, y)| \leq \frac{C^+}{(\tau - s)^{\mu}|x - y|^{n+1-2\mu}}, \tag{234}$$

for some $\mu \in (0.5, 1)$ and with some constant C^+. This is locally integrable with respect to time and space. This implies that there exists $C_\Gamma > 0$ such that

$$\sup_{(\tau,x) \in D^\tau_l} \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \sup_{\mathbf{f} \in S_0} |\Gamma^{l,*}_{f}| dy ds \leq C_\Gamma, \tag{235}$$

and such that

$$\sup_{(\tau,x) \in D^\tau_l} \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \sup_{\mathbf{f} \in S_0} |\Gamma^{l,*}_{f,i}| dy ds \leq C_\Gamma, \tag{236}$$

where $1 \leq i \leq n$. Hence, since $\mathbf{g} \in S_0$, for the second order derivatives of the first summand (208) we get

$$\begin{array}{l}
|v^{f,\rho,l,1}_{i,k,m} - v^{g,\rho,l,1}_{i,k,m}|_0 \leq \\
\\
\rho_l \int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \sum_{j=1}^n |f_j - g_j|_0 |\mathbf{v}^{g,\rho,l}|_{0,2} C_\Gamma \\
\\
+\rho_l \int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \sum_{j=1}^n |f_{j,m} - g_{j,m}|_0 |\mathbf{v}^{g,\rho,l}|_{0,1} C_\Gamma \\
\\
\leq \rho_l \int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} |\mathbf{f} - \mathbf{g}|^n_{0,1} |\mathbf{v}^{g,\rho,l}|^n_{0,2} C_\Gamma \\
\\
\leq \rho_l C^{l-1}_{1,2} C^2_\Gamma |\mathbf{f} - \mathbf{g}|^n_{0,1}.
\end{array} \tag{237}$$

Note that in this estimate the second C_Γ comes from the estimate of $|\mathbf{v}^{g,\rho,l}|^n_{0,2}$. From equation (220) we get

$$
\begin{array}{l}
|v^{f,\rho,l,2}_{i,l,m} - v^{g,\rho,l,2}_{i,l,m}|_0 := \\
\\
\leq \int_{(l-1)}^{\tau} \rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{n,m}(z-y)\times \\
\\
\left(\left(\sum_{j,k=1}^n |f_{k,j,i} + g_{k,j,i}|_0 \right) \times \right. \\
\\
\left. \left(\sum_{j,k=1}^n |f_{j,k,i} - g_{j,k,i}|_0 \right) \right) C_\Gamma \\
\\
\leq \rho_l C_K n^2 4 C_{1,2}^{l-1} C_\Gamma |\mathbf{f}-\mathbf{g}|^n_{0,2}.
\end{array}
\tag{238}
$$

Here the constant C_K is from the spatial integral with integrand $K_{n,m}$. We may assume without loss of generality that $C_\Gamma \geq 1$. The constant C_K depends on dimension only and may be subsumed by a constant C_n^*. In the following we shall use the latter constant generically if we want to subsume all constants depending only on dimension. We summarize that equation (237) and (238) gives

$$
\sum_{j,m=1}^n |v^{f,\rho,l}_{i,j,m} - v^{g,\rho,l}_{i,j,m}|_0 \leq \rho_l C_n^* C_\Gamma^2 C_{1,2}^{l-1} |\mathbf{f}-\mathbf{g}|^n_{0,2}. \tag{239}
$$

Similarly we get

$$
\sum_{m=1}^n |v^{f,\rho,l}_{i,m} - v^{g,\rho,l}_{i,m}|_0 \leq \rho_l C_n^* C_\Gamma^2 C_{1,2}^{l-1} |\mathbf{f}-\mathbf{g}|^n_{0,2}. \tag{240}
$$

and

$$
|v^{f,\rho,l}_{i} - v^{g,\rho,l}_{i}|_0 \leq \rho_l C_n^* C_\Gamma^2 C_{1,2}^{l-1} |\mathbf{f}-\mathbf{g}|^n_{0,2}. \tag{241}
$$

An estimate for the time derivatives of $v_i^{f,\rho,l}$ and $v_i^{g,\rho,l}$ can be reduced to the estimates (239), (240), and (241) via the defining equation for Γ^l_f

$$
\frac{\partial \Gamma^l_f}{\partial \tau} = \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \Gamma^l_f}{\partial x_j^2} - \rho_l \sum_{j=1}^n f_j \frac{\partial \Gamma^l_f}{\partial x_j} = 0. \tag{242}
$$

Remark 2.16. Alternatively we could use defining equations

$$
\begin{array}{l}
\frac{\partial v_i^{f,\rho,l}}{\partial \tau} = \rho_l \nu \sum_{j=1}^n \frac{\partial^2 v_i^{f,\rho,l}}{\partial x_j^2} - \rho_l \sum_{j=1}^n f_j \frac{\partial v_i^{f,\rho,l}}{\partial x_j} + \\
\\
\int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(x-y) \right) \sum_{j,k=1}^n \left(\frac{\partial f_k}{\partial x_j} \frac{\partial f_j}{\partial x_k} \right) (\tau, y) dy
\end{array}
\tag{243}
$$

for the functions $v_i^{f,\rho,l}$ and analogous defining equations for the functions $v_i^{g,\rho,l}$. However this would enlarge the estimate constant. In this case we get another factor C_K and another factor $C^l_{1,2}$. The method of growth control described in step 3 still applies.

Summarizing we get

$$
\begin{aligned}
&|\mathbf{v}^{f,\rho,l}-\mathbf{v}^{g,\rho,l}|^n_{1,2}=\sum_{i=1}^n|v_i^{f,\rho,l}-v_i^{g,\rho,l}|_{1,2}\\
&\leq \rho_l C_n^* C_\Gamma^2 C_{1,2}^{l-1}|\mathbf{f}-\mathbf{g}|^n_{0,2}\\
&\leq \rho_l C_n^* C_\Gamma^2 C_{1,2}^{l-1}|\mathbf{f}-\mathbf{g}|^n_{1,2}.
\end{aligned}
\tag{244}
$$

for some generic constant $C_n^*>0$ depending only on dimension n. We may choose

$$
\rho_l\leq \frac{1}{4C_n^*C_\Gamma^2C_{1,2}^{l-1}}.
\tag{245}
$$

Then for $\mathbf{f},\mathbf{g}\in S_0$ we have

$$
|\mathbf{v}^{f,\rho,l}-\mathbf{v}^{g,\rho,l}|^n_{1,2}\leq \tfrac{1}{4}|\mathbf{f}-\mathbf{g}|^n_{1,2}.
\tag{246}
$$

Note that

$$
\mathbf{f},\mathbf{g}\in S_0\rightarrow \mathbf{v}^{f,\rho,l}\in S_0
\tag{247}
$$

for this choice of ρ_l. Furthermore, note that the constant C_Γ depends on the class S_0. Hence, we do not have a contraction on the whole space but on a subspace S_0 of $C_b^{1,2}$. Since the definition of S_0 contains a factor 2 and we start with $|v^{\rho,l-1}|\leq C_{1,2}^{l-1}$ the iterations are $F_l^k(\mathbf{v}^{\rho,l-1}),\ k\geq 1$ belong to S_0 indeed by choice of ρ_l. Hence, the contraction condition leads to an iteration in a subspace S_0 of $C_b^{1,2}$. However, note that this is not a Banach space. In order to have the functional series in a Banach space or in order to have the limit of the functional series $\mathbf{v}^{\rho,l}=\lim_{k\uparrow\infty}\mathbf{v}^{\rho,k,l}$ known to be in some regular space, we need some decay at spatial infinity. Let us summarize first what we have achieved so far on the way of the construction of a local solution. Iteration of the map F_l starting with the function $\mathbf{v}^{\rho,l-1}$ leads to a series of functions $\mathbf{v}^{\rho,k,l}=F_l^{k+1}\left(\mathbf{v}^{\rho,l-1}\right)$ where the components of the functions $\mathbf{v}^{\rho,k,l}$ satisfy

$$
v_i^{\rho,k,l}=v_i^{\rho,0,l}+\sum_{m=1}^k\delta v_i^{\rho,m,l}\in C_b^{1,2},
\tag{248}
$$

and where the increments $\delta v_i^{\rho,m,l}$ satisfy a contraction property

$$
\begin{aligned}
&|\delta\mathbf{v}^{\rho,k,l}|^n_{1,2}\leq \rho_l C_n^*C_\Gamma^2C_{1,2}^{l-1}|\delta\mathbf{v}^{\rho,k-1,l}|^n_{1,2}\\
&\leq \tfrac{1}{4}|\delta\mathbf{v}^{\rho,k-1,l}|^n_{1,2}.
\end{aligned}
\tag{249}
$$

The approximations $v_i^{\rho,k,l}$ can be represented in terms of fundamental solution Γ_k^l, where for $k\geq 1$ Γ_k^l is the fundamental solution of the equation

$$
\frac{\partial\Gamma_k^l}{\partial\tau}-\rho_l\nu\Delta\Gamma_k^l+\rho_l\sum_{j=1}^n v_j^{\rho,k-1,l}\frac{\partial\Gamma_k^l}{\partial x_j}=0,
\tag{250}
$$

and for $k=0$ the function Γ^l_0 is the fundamental solution of the equation

$$\frac{\partial \Gamma^l_0}{\partial \tau}-\rho_l\nu\Delta\Gamma^l_0+\rho_l\sum_{j=1}^n v^{\rho,l-1}_j\frac{\partial \Gamma^l_0}{\partial x_j}=0. \tag{251}$$

The choice of ρ_l ensures that the first order coefficients in (250) and (251) have a uniform bound. This leads to the estimate of integrals involving integrands Γ_k by a uniform constant C_Γ such that

$$|\mathbf{v}^{\rho,k,l}|^n_{1,2}=|\mathbf{v}^{\rho,l-1}+\sum_{m=1}^k\delta\mathbf{v}^{\rho,m,l}|^n_{1,2}\leq 2C^{l-1}_{1,2}, \tag{252}$$

and, hence, to a functional series in S_0. This leads to the contraction property of the series $F^k_l\left(\mathbf{v}^{\rho,l-1}\right)$ with respect to the norm $|.|_{1,2}$, and we have (193) especially.

Next in order to apply method a) of the introduction we show that the elements of the functional series $v^{\rho,k,l}_i(\tau,.)$ are in H^2 uniformly with respect to $\tau\in[l-1,l]$ and for all $1\leq i\leq n$. This means that we extend the estimate (249) above to the space of functions $\mathbf{f}$ with components

$$f_i\in H^2_l:=\left\{h\in C^{1,2}_b(D^\tau_l)|h(\tau,.)\in H^2\ \text{ unif. for all } \tau\in[l-1,l]\right\}. \tag{253}$$

Here 'unif.' is an abbreviation for uniformly. This is a good space in order to have a finite magnitude (188), since we have

$$\begin{array}{l}\int_{\mathbb{R}^n}\sum_{j,k=1}^n\left|\left(\frac{\partial v^{\rho,l}_k}{\partial x_j}\frac{\partial v^{\rho,l}_j}{\partial x_k}\right)(t,y)\right|dy\\ \int_{\mathbb{R}^n}\sum_{j,k=1}^n\left|\left(\frac{1}{2}\left(\frac{\partial v^{\rho,l}_k}{\partial x_j}\right)^2+\frac{1}{2}\left(\frac{\partial v^{\rho,l}_j}{\partial x_k}\right)\right)^2(t,y)\right|dy<\infty.\end{array} \tag{254}$$

if $v^{\rho,l}_i\in H^1$. We choose the space H^2 since we have first derivatives of (188) involved in our estimation.

Recall that method a) of the introduction applies in the case $n=3$. Note that a converging series

$$\mathbf{v}^{\rho,k,l}(\tau,.)\in\left[H^s\left(\mathbb{R}^n\right)\right]^n\quad\text{for some }\ s\geq\alpha+\frac{1}{2}n \tag{255}$$

(uniformly in τ) implies that

$$\mathbf{v}^{\rho,l}(\tau,.)\in\left[H^s\left(\mathbb{R}^n\right)\right]^n\quad\text{for some }\ s\geq\alpha+\frac{3}{2}. \tag{256}$$

Hence, in case $n=3$ we have

$$\mathbf{v}^{\rho,l}(\tau,.)\in\left[C^\alpha\left(\mathbb{R}^n\right)\right]^n \tag{257}$$

for $\alpha \in (0, 0.5)$ uniformly in τ. Hence, if we can show that (255) holds (which implies (256), then we have bounded Hölder continuous first order coefficients in the equation

$$\frac{\partial \Gamma^l_v}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \Gamma^l_v}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{\rho,l} \frac{\partial \Gamma^l_v}{\partial x_j} = 0, \tag{258}$$

and this means that a fundamental solution Γ^l_v of (258) exists. Then we have the representation

$$\begin{array}{l} v_i^{\rho,l}(\tau,x) = \int_{\mathbb{R}^n} v_i^{\rho,l-1}(l-1,y)\Gamma^l_v(\tau,x;l-1,y)dy+ \\ \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(y-z)\right) \left(\sum_{j,k=1}^n \left(\frac{\partial v_k^{\rho,l}}{\partial x_j} \frac{\partial v_j^{\rho,l}}{\partial x_k}\right)(\tau,z)\right) \times \\ \times \Gamma^l_v(\tau,x,s,y)dzdyds \end{array} \tag{259}$$

and from this representation we immediately get

$$\mathbf{v}^{\rho,l} \in \left[C_b^{1,2}\left(D_l^{\tau}\right)\right]^n. \tag{260}$$

Now let us finish the proof in the case $n = 3$ using method a) of the introduction. We have to show that

$$v_i^{\rho,k,l}(\tau,.) \in H^2(\mathbb{R}^n), \tag{261}$$

where we show that the latter relation holds uniformly in τ. First, inductively with respect to l we have

$$v_i^{\rho,l-1}(l-1,.) \in C_b^2 \ \text{ and } \ v_i^{\rho,l-1}(l-1,.) \in H^2(\mathbb{R}^n). \tag{262}$$

First we show that

$$v_i^{\rho,k,l}(\tau,.) \in L^2 \tag{263}$$

uniformly in τ. The reasoning for the first and second derivatives below requires only a little more work. We start with $k = 0$. Note that the equation which defines $v_i^{\rho,0,l}$ has first order coefficient functions $v_i^{\rho,l-1}(l-1,.)$ which are independent of time. Hence the fundamental solution Γ^l_0 of the equation

$$\frac{\partial \Gamma^l_0}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \Gamma^l_0}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{\rho,l-1} \frac{\partial \Gamma^l_0}{\partial x_j} = 0 \tag{264}$$

exists. For this fundamental solution we have a priori estimates. Especially, we have

$$|\Gamma^l_0(\tau,x;s,y)| \leq \frac{C}{(\tau-s)^{n/2}} \exp\left(-\lambda \frac{|x-y|^2}{\tau-s}\right) \tag{265}$$

for some constants C and λ (the constants λ and C are used generically here). Next we have the representation

$$\begin{array}{ll}v^{\rho,0,l}_i(\tau,x)=\int_{\mathbb{R}^n}v^{\rho,l-1}_i(l-1,y)\Gamma^l_0(\tau,x;l-1,y)dy+\\ \int_{(l-1)}^{\tau}\int_{\mathbb{R}^n}\rho_l\int_{\mathbb{R}^n}\left(K_{n,i}(y-z)\right)\left(\sum_{j,k=1}^n\left(\frac{\partial v^{\rho,l-1}_k}{\partial x_j}\frac{\partial v^{\rho,l-1}_j}{\partial x_k}\right)(\tau,z)\right)\times\\ \times\Gamma^l_0(\tau,x,s,y)dzdyds\end{array}\tag{266}$$

Note that for $\tau=l-1$ we have $v^{\rho,l-1}(l-1,.)\in H^2$ inductively. We get

$$\begin{array}{ll}|v^{\rho,0,l}_i(\tau,x)|\leq\int_{\mathbb{R}^n}|v^{\rho,l-1}_i(l-1,y)|\frac{C}{(\tau-(l-1))^{n/2}}\exp\left(-\lambda\frac{|x-y|^2}{\tau-(l-1)}\right)dy+\\ \int_{(l-1)}^{\tau}\int_{\mathbb{R}^n}\rho_l\Big|\int_{\mathbb{R}^n}\left(K_{n,i}(y-z)\right)\left(\sum_{j,k=1}^n\left(\frac{\partial v^{\rho,l-1}_k}{\partial x_j}\frac{\partial v^{\rho,l-1}_j}{\partial x_k}\right)(s,z)\right)\Big|\times\\ \times\frac{C}{(\tau-s)^{n/2}}\exp\left(-\lambda\frac{|x-y|^2}{\tau-s}\right)dzdyds\\ =:(I)+(II).\end{array}\tag{267}$$

Now (I) and (II) are convolutions. We may apply Young's inequality, i.e.,

$$|f\star g|_r\leq|f|_q|g|_p,\tag{268}$$

where

$$1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q},\tag{269}$$

and $1\leq p,q,r\leq\infty$. Note that the family of Gaussian functions

$$x\rightarrow\frac{C}{(\tau-s)^{n/2}}\exp\left(-\lambda\frac{|x|^2}{\tau-s}\right)\tag{270}$$

(family with respect to time parameters) is in L^p for all $1\leq p<\infty$ (although not in L^∞ as $s\uparrow\tau$). We know that the first term (I) is in L^2 because $v^{\rho,l-1}_i\in L^2$ inductively and the Gaussian is in L^1. Mote that we may set the function in (270) to zero if $\tau=s$. Furthermore, applying Young's inequality for fixed $t>s$ first we see that the integrand for the second term is in L^2 where $v^{\rho,l-1}\in H^2$ inductively. Since the L^2-bound of the integrand is uniform the time integral (II) is also L^2. Here again, for each $x\in\mathbb{R}^n$ you may use decompose the integral in two parts $[0,\tau]\times B_x$ and $[0,\tau]\times\mathbb{R}^n\setminus([0,\tau]\times B_x)$, where B_x is a n-dimensional ball around x of finite radius. Then on the ball we may use the estimate

$$\frac{1}{(\tau-s)^{n/2}}\exp\left(-\lambda\frac{|x-y|^2}{\tau-s}\right)\leq\frac{C}{(t-s)^{\mu}|x-y|^{n-2\mu}}\tag{271}$$

for $\mu \in (0,1)$. Outside the ball we may use the estimate

$$\left|\Gamma_0^l(\tau,x;s,y)\right| \leq \frac{C}{(\tau-s)^{n/2}} \exp\left(-\lambda \frac{|x-y|^2}{\tau-s}\right). \tag{272}$$

Hence we have

$$v_i^{\rho,0,l}(\tau,.) \in L^2(\mathbb{R}^n) \tag{273}$$

uniformly in τ. Next note that we have representations of the first and second spatial derivatives of $v_i^{\rho,0,l}$ which involve only the first spatial derivative of the fundamental solution (for the second spatial derivative we use the adjoint and partial integration). Then an analogous argument as above which uses the first derivative estimates

$$\frac{\partial}{\partial x_i} \frac{1}{(\tau-s)^{n/2}} \exp\left(-\lambda \frac{|x-y|^2}{\tau-s}\right) \leq \frac{C}{(t-s)^{\mu}|x-y|^{n+1-2\mu}} \tag{274}$$

for $\mu \in (0.5,1)$, and

$$\left|\Gamma_0^l(\tau,x;s,y)\right| \leq \frac{C}{(\tau-s)^{(n+1)/2}} \exp\left(-\lambda \frac{|x-y|^2}{\tau-s}\right). \tag{275}$$

leads to the conclusion that

$$v_i^{\rho,0,l}(\tau,.) \in H^2(\mathbb{R}^n) \tag{276}$$

uniformly in τ. Next assuming that

$$v_i^{\rho,k-1,l} \in C_b^{1,2} \text{ and } v_i^{\rho,k-1,l}(\tau,.) \in H^2(\mathbb{R}^n) \tag{277}$$

uniformly in $\tau \in [l-1,l]$ we know that $v_i^{\rho,k,l} \in C_b^{1,2}$ and we want to show that

$$v_i^{\rho,k,l}(\tau,.) \in H^2(\mathbb{R}^n) \tag{278}$$

uniformly in τ. It is convenient to use the series $\left(\mathbf{v}^{\rho,k,l}\right)_k$ along with

$$\mathbf{v}^{\rho,k,l} = \mathbf{v}^{\rho,0,l} + \sum_{m=1}^{k} \delta\mathbf{v}^{\rho,m,l}. \tag{279}$$

Then it suffices to show a a contraction property with respect to the H^2-norm in the sense that

$$|\delta v_i^{\rho,k,l}(\tau,.)|_{H^2} \leq \frac{1}{4}|\delta v_i^{\rho,k-1,l}(\tau,.)|_{H^2} \tag{280}$$

for all $k \geq 1$. This holds if we use a constant ρ_l with

$$\rho_l \leq \frac{1}{4C_n^* C_\Gamma^2 (C_{1,2}^{l-1} + lC_{1,2}^{l-1})}. \tag{281}$$

We consider the main steps in order ro prove this relation. Note that the equation which defines $v_i^{\rho,k,l}$ has first order coefficient functions $v_i^{\rho,k-1,l} \in C_b^{1,2}$. Hence the fundamental solution Γ_k^l of the equation

$$\frac{\partial \Gamma_k^l}{\partial \tau} - \rho_l \nu \sum_{j=1}^n \frac{\partial^2 \Gamma_k^l}{\partial x_j^2} + \rho_l \sum_{j=1}^n v_j^{\rho,k-1,l} \frac{\partial \Gamma_k^l}{\partial x_j} = 0 \tag{282}$$

exists. Since we have $|v_j^{\rho,k-1,l}|_{1,2} \leq 2C_{1,2}^l$ independent of the number k. For this fundamental solution we have a priori estimates. Especially, we have

$$|\Gamma_k^l(\tau, x; s, y)| \leq \frac{C}{(\tau - s)^{n/2}} \exp\left(-\lambda \frac{|x-y|^2}{\tau - s}\right) \tag{283}$$

for some $\lambda, C > 0$ independent of the number k. Next we have the representation

$$\begin{array}{l}
\delta v_i^{\rho,k,l}(\tau, x) = -\rho_l \int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \left(\sum_{j=1}^n \delta v_j^{\rho,k-1,l} \frac{\partial v^{\rho,k-1,l}}{\partial x_j} \right)(s,y) \Gamma_k^l(\tau, x; s, y) ds dy + \\
\int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(y-z) \right) \left(\sum_{j,m=1}^n \left(\frac{\partial v_m^{\rho,k-1,l}}{\partial x_j} \frac{\partial v_j^{\rho,k-1,l}}{\partial x_m} \right)(\tau, z) \right) \times \\
\times \Gamma_k^l(\tau, x, s, y) dz dy ds \\
-\int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \rho_l \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i} K_n(y-z) \right) \left(\sum_{j,m=1}^n \left(\frac{\partial v_m^{\rho,k-2,l}}{\partial x_j} \frac{\partial v_j^{\rho,k-2,l}}{\partial x_m} \right)(\tau, z) \right) \times \\
\times \Gamma_k^l(\tau, x, s, y) dz dy ds
\end{array} \tag{284}$$

This leads to

$$\begin{array}{l}
\delta v_i^{\rho,k,l}(\tau, x) = -\rho_l \int_{(l-1)}^{\tau} \int_{\mathbb{R}^n} \left(\sum_{j=1}^n \delta v_j^{\rho,k-1,l} \frac{\partial v^{\rho,k-1,l}}{\partial x_j} \right)(s,y) \Gamma_k^l(\tau, x; s, y) ds dy + \\
\int_{(l-1)}^{\tau} \rho_l \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{n,i}(z-y) \times \\
\left(\left(\sum_{j,m=1}^n \left(v_{m,j}^{\rho,k-1,l}(s,y) + v_{m,j}^{\rho,k-2,l}(s,y) \right) \right) \times \right. \\
\left. \left(\sum_{j,m=1}^n \left(v_{j,m}^{\rho,k-1,l}(s,y) - v_{j,m}^{\rho,k-2,l}(s,y) \right) \right) \right) \Gamma_k^l(\tau, x; s, z) dy dz ds,
\end{array} \tag{285}$$

Hence, using a priori estimates analogously as above in case $k = 0$ we get (280). This finishes the proof in the case $n = 3$. The method a) as it is presented above applies in the case $n = 3$. In case $n > 3$ we have to ensure that the convergence of the series $v_i^{\rho,k,l}(\tau, .)$ is in $H^{s,p}$ for p large enough. This can be done using the same a priori estimates. Method b) provides an alternative which may be even more elementary. This direct

(more elementary) way shows that the limit of the functional series exists in a Banach space by transforming on a compact domain. Such a transformation can be applied if for each $k\geq 0$ we have

$$|\partial^{\alpha}_x \mathbf{v}^{\rho,k,l}(t,x)|\leq \frac{C_{\alpha k}}{(1+|x|)^5} \text{ if } |\alpha|\leq 2 \quad . \tag{286}$$

for $|\alpha|\leq 2$ and $k\leq 5$. Then we may consider the transformation of spatial variables

$$x_i\rightarrow y_i:=\arctan\left(x_i\right) \tag{287}$$

from $\mathbb{R}^n$ to $\left]-\frac{\pi}{2},\frac{\pi}{2}\right[^n$ and consider the transformed function

$$w^{f,\rho,l}_i(\tau,y):=v^{f,\rho,l}_i(\tau,x) \tag{288}$$

for $1\leq i\leq n$. We have

$$\frac{\partial v^{f,\rho,l}_i}{\partial x_m}=\frac{\partial w^{f,\rho,l}_i}{\partial y_m}\frac{\partial y_m}{\partial x_m}=\frac{\partial w^{f,\rho,l}_i}{\partial y_m}\frac{1}{1+x_m^2}=\frac{\partial w^{f,\rho,l}_i}{\partial y_m}\frac{1}{1+\tan^2(y_m)}, \tag{289}$$

and

$$\begin{array}{l}\frac{\partial^2 v^{f,\rho,l}_i}{\partial x^2_m}=\frac{\partial^2 w^{f,\rho,l}_i}{\partial y^2_m}\left(\frac{\partial y_m}{\partial x_m}\right)^2+\frac{\partial w^{f,\rho,l}_i}{\partial y_m}\frac{\partial^2 y_m}{\partial x^2_m}=\\ \\ \frac{\partial^2 w^{f,\rho,l}_i}{\partial y^2_m}\left(\frac{1}{1+\tan^2(y_m)}\right)^2+\frac{\partial w^{f,\rho,l}_i}{\partial y_m}\frac{-2\tan(y_m)}{(1+\tan^2(y_m))^4}.\end{array} \tag{290}$$

Since

$$\left(1+|x|^4\right)\frac{\partial^2 v^{f,\rho,l}_i}{\partial x^2_m}\downarrow 0 \text{ as } |x|\uparrow\infty \tag{291}$$

Hence we get a series of functions $\left(\mathbf{w}^{\rho,k,l}\right)_k$ defined on transformed domain $D^{\tau,\pi}_l:=[l-1,l]\times\left]-\pi/2,\pi/2\right[^n$ along with

$$\mathbf{w}^{\rho,l,k}(\tau,y)=\mathbf{v}^{\rho,l,k}(\tau,x) \tag{292}$$

and the limit is in the Banach space $\left[C_{1+\alpha/2,2+\alpha}\left(D^{\tau,\pi}_l\right)\right]^n$ with the norm $|.|^n_l$ restricted to the domain $D^{\tau,\pi}_l$. Note that we have $w^{\rho,k,l}_i(\tau,\mathbf{s})=0$ for all $1\leq i\leq n$ if for some s_j in $\mathbf{s}=(s_1,\cdots s_n)$ we have $s_j\in\{-\pi/2,\pi/2\}$ according to the polynomial decay of $\mathbf{v}^{\rho,k,l}$ at infinity. Hence we get a limit $\mathbf{w}^{\rho,l}$ with $\mathbf{w}^{\rho,l}(\tau,y)=\mathbf{v}^{\rho,l}(\tau,x)$ for all $(\tau,x)\in D^{\tau}_l$ and $(\tau,y)\in D^{\tau,\pi}_l$ where satisfies the transformed Navier-Stokes equation (in τ,y-coordinates on $D^{\tau,\pi}_l$).

2.2 step 2: Extension of local existence to extended equations for $\mathbf{v}^{r,\rho,l}$

In step 1 we did not consider the growth of the solution. We just proved that the iterations $F^m_l(\mathbf{v}^{\rho,l-1})$, $l\geq 1$ of the functional F_l are in a function

set S_0 such that first order terms of the linear subproblems are uniformly bounded. In order to get a globally bounded scheme we introduce a control function $\mathbf{r} = (\mathbf{r}^l)_l$ and set up a globally bounded scheme for $\mathbf{v}^{r,\rho} = \mathbf{v}^\rho + \mathbf{r}$. In this second step we observe that the extension of the local iteration of step 1 to equations with bounded regular functions $\mathbf{r} \neq 0$ does not change the argument of step 1 essentially. Note that the Leray projection from equation for $\mathbf{v}^{r,\rho}$ includes integral terms for r_i^l as described in the introduction. This means that we have to estimate magnitudes of the from

$$\int_{\mathbb{R}^n} \sum_{j,k,m,p=1}^{n} \Big|\frac{\partial r_k^l}{\partial x_j}\Big|\Big|\frac{\partial r_m^l}{\partial x_p}\Big|(\tau, y)dy. \tag{293}$$

Again this reduces to H^2 estimates. The similarity of the convergence argument of the local scheme for $\mathbf{v}^{r,\rho,l}$ to the argument for convergence of the local scheme of step 1 is due to the fact that the additional terms symbolized by the term $L_i^{\rho,l}(\mathbf{r}, \mathbf{v})$ in the equation for $\mathbf{v}^{r,\rho,l}$ consist of a source term dependent on the function $\mathbf{r}$ (which we called $S^l(\mathbf{r})$), and a linear operator dependent on the function $\mathbf{v}^{r,\rho,l}$ which we called $L_i^{\rho,l,0}(\mathbf{r}, \mathbf{v})$ (resp. $\mathbf{v}^{r,\rho,k,l}$ for each step of the local iteration). Note that both additional terms have a multiplier ρ_l with the exception of the time derivative of $r_{i,\tau}^l$. If $\mathbf{r}^l$ is regular for given time step number l, i.e., if the first time derivative and the second time derivatives exist and are integrable on the domain $(l-1, l] \times \mathbb{R}^n$, then we get a regular solution $\mathbf{v}^{r,\rho,0,l}$ of the linearized Navier-Stokes equation at the first step of the local time iteration. Furthermore, the additional source term $S^l(\mathbf{r})$ depending on the function $\mathbf{r}$ appears only in the equation for $\mathbf{v}^{r,\rho,0,l}$, because these source terms cancel out in the subtraction $\delta\mathbf{v}^{r,\rho,k,l} = \mathbf{v}^{r,\rho,k,l} - \mathbf{v}^{r,\rho,k-1,l}$ defining the higher order corrections of the local iterative scheme. It is sufficient to find a lower bound $\rho_l \gtrsim \frac{1}{l}$ for the number ρ_l in order to make the scheme global.

Remark 2.17. For numerical purposes we shall consider a different solution scheme below. If we know by analytical means that $\mathbf{v}^r$ and $\mathbf{r}$ are bounded then we know that $\mathbf{v}$ is bounded, and this implies that we can set up a scheme where we do not use the function $\mathbf{r}$ at all. A second feature of a numerical scheme is that diffusion may imply that increasing time step sizes ρ_l may be chosen as the time step number l increases. This makes the scheme more efficient.

Next we construct

$$\mathbf{v}^{r,\rho,l} = \mathbf{v}^{r,\rho,0,l} + \sum_{k=1}^{\infty} \delta\mathbf{v}^{r,\rho,k,l}. \tag{294}$$

In this section our goal is make the extension of the argument in step 1 to the case where $\mathbf{r} \neq 0$ as easy as possible. Therefore we write down an iterative scheme for (294) which we would not choose for numerical purposes. More

precisely, for each $k \geq 0$ in the computation of the series (294) we have the additional term $L_i^{r,\rho,l}$ with arguments of the previous time step. This means that in step $k=0$ we add a term $L_i^{r,\rho,l}\left(\mathbf{r},\mathbf{v}^{\rho,l-1}\right)$ on the right side in order to compute the approximation $v_i^{r,\rho,0,l}$ (note that the argument $\mathbf{r}$ is considered to be externally given in this second step of the proof; we shall determine the function $\mathbf{r}$ in the third step if this proof). Similarly, in step $k>0$ we add a term $L_i^{r,\rho,l}\left(\mathbf{r},\mathbf{v}^{\rho,k-1,l}\right)$ on the right side in order to compute the approximation $v_i^{r,\rho,k,l}$. The first term of this functional series $\mathbf{v}^{r,\rho,0,l}=\left(v_1^{r,\rho,0,l},\cdots,v_n^{r,\rho,0,l}\right)^T$ is solution of the equation (for $1\leq i\leq n$)

$$\left\lbrace\begin{array}{ll}
\frac{\partial v_i^{r,\rho,0,l}}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v_i^{r,\rho,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j}=L_i^{r,\rho,l}\left(\mathbf{r},\mathbf{v}^{r,\rho,l-1}\right)\\
\\
+\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,y)dy+r_{i,\tau}^l,\\
\\
\mathbf{v}^{r,\rho,0,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.),
\end{array}\right. \tag{295}$$

and the other terms $\delta\mathbf{v}^{r,\rho,k,l}$ of this functional series are the respective solutions of the equations (for $1\leq i\leq n$)

$$\left\lbrace\begin{array}{ll}
\frac{\partial\delta v_i^{r,\rho,k,l}}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2\delta v_i^{r,\rho,k,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,k-1,l}\frac{\partial\delta v_i^{r,\rho,k,l}}{\partial x_j}=L_i^{\rho,l,0}\left(\mathbf{r},\mathbf{v}^{r,\rho,k-1,l}\right)\\
\\
-\rho_l\sum_{j=1}^n\delta v_j^{r,\rho,k-1,l}\frac{\partial v_i^{r,\rho,k-1,l}}{\partial x_j}+\\
\\
\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v_m^{r,\rho,k-1,l}}{\partial x_j}\frac{\partial v_j^{r,\rho,k-1,l}}{\partial x_m}\right)(\tau,y)dy\\
\\
-\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,k-2,l}}{\partial x_j}\frac{\partial v_j^{r,\rho,k-2,l}}{\partial x_k}\right)(\tau,y)dy,\\
\\
\delta\mathbf{v}^{r,\rho,k,l}(l-1,.)=0.
\end{array}\right. \tag{296}$$

Here in case $k=1$ the expression $v_j^{r,\rho,k-2,l}$ is defined by

$$v_j^{r,\rho,-1,l}:=v_j^{r,\rho,l-1}. \tag{297}$$

We assume that from the previous time step $l-1$ we have the estimates

$$|r_i^{l-1}|_0\leq C_r^0,\ |r_i^{l-1}|_{0,1}\leq C_r^1,\ |r_i^{l-1}|_{1,2}\leq C_r, \tag{298}$$

and that

$$|v_i^{l-1}|_0\leq C_0^{l-1},\ |v_i^{l-1}|_{0,1}\leq C_1^{l-1},\ \sum_{i=1}^n|v_i^{l-1}|_{1,2}\leq C_{1,2}^{l-1}. \tag{299}$$

Furthermore we assume that

$$\left|v_k^{r,\rho,l-1}(\tau,.)\right|_{H^2}\leq C_n^*\left(C_{1,2}^{l-1}+(l-1)C_{1,2}^{l-1}\right), \tag{300}$$

and that

$$\left|r_k^{l-1}\right|_{H^2}\leq C_n^*\left(C_r+(l-1)C_r\right) \tag{301}$$

for all $1\leq k\leq n$ and generic C_n^*. Note that this implies

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n\Big|\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j}\Big|\Big|\frac{\partial v_m^{r,\rho,l-1}}{\partial x_p}\Big|(\tau,y)dy\leq C_n^*\left(C_{1,2}^{l-1}+(l-1)C_{1,2}^{l-1}\right), \tag{302}$$

and that

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n\Big|\frac{\partial r_k^{l-1}}{\partial x_j}\Big|\Big|\frac{\partial r_m^{l-1}}{\partial x_p}\Big|(\tau,y)dy\leq C_n^*\left(C_r+(l-1)C_r\right). \tag{303}$$

If we establish the latter estimates for all l then we have a linear growth of the integral terms of Leray projection in the equations for $\mathbf{v}^{r,\rho,l}$.

Remark 2.18. In the next subsection we shall define $\mathbf{r}^l$ and determine the constant C_r. We shall show that there exists a constant $C_r>0$ (independent of l) such that for all $1\leq i\leq n$ the functions r_i^l satisfy

$$|r_i^l|_{1,2}\leq C_r \tag{304}$$

uniformly, i.e., independent of the time step number l. Here the norm is defined on the local domain $D_l^\tau=(l-1,l]\times\mathbb{R}^n$, i.e. we have

$$|r_i^l|_{1,2}:=\sum_{|\alpha|\leq 2}\sup_{(\tau,x)\in D_l^\tau}|r^l_{,\alpha}(\tau,x)|+\sup_{(\tau,x)\in D_l^\tau}|r^l_{,\tau}(\tau,x)| \tag{305}$$

for multiindices α related to the spatial variables (especially, we have $|r^l_{i,\tau}|\leq C_r$ for all $1\leq i\leq n$).

In step 3 of this proof below we construct r_i^l such that the relation (304) and the relation (303) are satisfied for all $l\geq 1$. Next recall that

$$\begin{array}{l}L_i^{\rho,l}(\mathbf{r};\mathbf{v}^{r,\rho,l-1})\equiv-\rho_l\nu\Delta r_i^l+\rho_l\sum_{j=1}^n r_j^l\frac{\partial r_i^l}{\partial x_j}\\ \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^l}{\partial x_j}\frac{\partial r_j^l}{\partial x_k}\right)(\tau,y)dy+L_i^{\rho,l,0}(\mathbf{r};\mathbf{v}^{r,\rho,l-1})\end{array} \tag{306}$$

where

$$\begin{array}{l}L_i^{\rho,l,0}(\mathbf{r};\mathbf{v})\equiv+\rho_l\sum_{j=1}^n r_j^l\frac{\partial v_i^{r,\rho,l}}{\partial x_j}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l}\frac{\partial r_i^l}{\partial x_j}\\ \\ -2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^l}{\partial x_j}\frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy\end{array} \tag{307}$$

We may use the rough estimate

$$|L_i^{\rho,l}(\mathbf{r};\mathbf{v}^{r,\rho,l-1})| \leq \rho_l \nu C_r + \rho_l n C_r^2 + \rho_l C_K C_r^2 + |L_i^{\rho,l,0}(\mathbf{r};\mathbf{v}^{r,\rho,l-1})|. \quad (308)$$

which we get from (304). This is not needed essentially for the proof of local convergence of this step. For this purpose we use (304) again, and we get from (307) the estimate (generic use of C_n^*)

$$|L_i^{\rho,l,0}(\mathbf{r};\mathbf{v}^{r,\rho,l-1})| \leq \rho_l C_n^* |\mathbf{r}^l|_{0,1} |\mathbf{v}^{r,\rho,l-1}|_{0,1} \quad (309)$$

For $l \geq 1$ given the functional series $(v_i^{r,\rho,k,l})_k$ is determined by the representations

$$\begin{array}{l}
v_i^{r,\rho,0,l}(\tau,x) = \int_{\mathbb{R}^n} v_i^{r,\rho,l-1}(l-1,y)\Gamma_0^l(\tau,x;l-1,y)dy + \\
\int_{(l-1)}^{\tau}\int_{\mathbb{R}^n}\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(y-z)\right)\left(\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,z)\right)\times \\
\times\Gamma_0^{r,l}(\tau,x,s,y)dzdyds \\
+\int_{(l-1)}^{\tau}\int_{\mathbb{R}^n} L_i^{\rho,l}(\mathbf{r},\mathbf{v}^{r,l-1})(s,y)\Gamma_0^{r,l}(\tau,x,s,y)dzdyds \\
+\int_{(l-1)}^{\tau}\int_{\mathbb{R}^n} r_{i,t}^l(s,y)\Gamma_0^{r,l}(\tau,x,s,y)dzdyds
\end{array} \quad (310)$$

and (for $k \geq 1$)

$$\begin{array}{l}
\delta v_i^{r,\rho,k,l}(\tau,x) = -\rho_l\int_{(l-1)}^{\tau}\int_{\mathbb{R}^n}\left(\sum_{j=1}^n \delta v_j^{r,\rho,k-1,l}\frac{\partial v^{r,\rho,k-1,l}}{\partial x_j}\right)(s,y)\Gamma_k^{r,l}(\tau,x;s,y)dsdy + \\
\int_{(l-1)}^{\tau}\rho_l\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}K_{n,i}(z-y)\times \\
\left(\left(\sum_{j,m=1}^n\left(v_{m,j}^{r,\rho,k-1,l}(s,y)+v_{m,j}^{r,\rho,k-2,l}(s,y)\right)\right)\times\right. \\
\left.\left(\sum_{j,m=1}^n\left(v_{j,m}^{r,\rho,k-1,l}(s,y)-v_{j,m}^{r,\rho,k-2,l}(s,y)\right)\right)\right)\Gamma_k^{r,l}(\tau,x;s,z)dydzds, \\
+\int_{(l-1)}^{\tau}\int_{\mathbb{R}^n} L_i^{\rho,l,0}\left(\mathbf{r},\delta\mathbf{v}^{r,\rho,k-1,l}\right)(s,y)\Gamma_k^{r,l}(\tau,x,s,y)dzdyds
\end{array} \quad (311)$$

where the functions $\Gamma_k^{r,l},\ k \geq 0$ are fundamental solutions of

$$\frac{\partial\Gamma_k^{r,l}}{\partial\tau} - \rho_l\nu\sum_{j=1}^n\frac{\partial^2\Gamma_k^{r,l}}{\partial x_j^2} + \rho_l\sum_{j=1}^n v_j^{r,\rho,k-1,l}\frac{\partial\Gamma_k^{r,l}}{\partial x_j} = 0 \quad (312)$$

(recall that $v_i^{r,\rho,-1,l} = v^{r,\rho,l-1}$ in case $k=0$). Analogously as in the first step of this proof for some time step l let $\mathbf{v}^{r,\rho,l-1}$ with $\mathbf{v}^{r,\rho,l-1}$ for some let $F_{r,l} : C_b^{1,2}(D_l^\tau)$ such that $\mathbf{v}^{r,\rho,k,l} = F_{r,l}^k(\mathbf{v}^{r,\rho,l-1})$. We assume that

$$|\mathbf{v}^{r,\rho,l-1}|_{1,2}^n \leq C_{1,2}^{l-1} \quad (313)$$

and start the iteration with

$$\mathbf{v}^{r,\rho,l-1}\in S_0^r:=\left\lbrace \mathbf{f}||\mathbf{f}|^n_{1,2}\leq 4C^{l-1}_{1,2}\right\rbrace .\tag{314}$$

Hence, similar as in the first step we get for some generic constant C^*_n dependent on the dimension n

$$\begin{array}{ll}
|\delta \mathbf{v}^{r,\rho,k,l}|^n_{1,2}\leq \rho_l C^*_n C_{\Gamma}\left(C^{l-1}_{1,2}+C_r\right)C_{\Gamma}|\delta\mathbf{v}^{\rho,k-1,l}|^n_{1,2}\\
\\
\leq \frac{1}{4}|\delta \mathbf{v}^{r,\rho,k-1,l}|^n_{1,2}
\end{array}\tag{315}$$

for all $k\geq 0$ if we choose

$$\rho_l\leq \frac{1}{4C^*_nC_{\Gamma}\left(C^{l-1}_{1,2}+C_r\right)C_{\Gamma}}\tag{316}$$

In order to have similar estimates with respect to the $|.|_{H^2}$-norm we use constants ρ_l which decrease linearly with respect to the time step number l, i.e. we use

$$\rho_l\leq \frac{1}{4C^*_nC_{\Gamma}\left(\left(C^{l-1}_{1,2}+C_r\right)+l\left(C^{l-1}_{1,2}+C_r\right)\right) C_{\Gamma}}\tag{317}$$

Moreover with analogous arguments as in the first step for all $\tau\in [l-1,l]$ we have

$$|\delta v^{r,\rho,k,l}(\tau,.)|_{H^2}\leq \frac{1}{4}|\delta v^{r,\rho,k-1,l}(\tau,.)|_{H^2}.\tag{318}$$

uniformly in τ. Then we use either method a) or b) completely analogously as in the first step of this proof. Hence locally and for ρ_l as above we have got a function $\mathbf{v}^{\rho,l}\in C^{1,2}_b$ which is constructed from a limit of the functional series

$$v^{r,\rho,l}_i=v^{r,\rho,0,l}_i+\sum_{k=1}^{\infty}\delta v^{r,\rho,k,l}_i\tag{319}$$

where $\mathbf{v}^{r,\rho,l}$ is a classical solution of

$$\left\lbrace \begin{array}{ll}
\frac{\partial v^{r,\rho,l}_i}{\partial \tau}-\rho_l\nu\sum_{j=1}^n \frac{\partial^2 v^{r,\rho,l}_i}{\partial x_j^2}+\rho_l\sum_{j=1}^n v^{r,\rho,l}_j\frac{\partial v^{r,\rho,l}_i}{\partial x_j}=\\
\\
L^{\rho,l}_i(\mathbf{r},\mathbf{v}^{r,\rho,l})+\\
\\
\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v^{r,\rho,l}_k}{\partial x_j}\frac{\partial v^{r,\rho,l}_j}{\partial x_k}\right)(\tau,y)dy+r^l_{i,\tau},\\
\\
\mathbf{v}^{r,\rho,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.).
\end{array}\right.\tag{320}$$

Finally, we note that by adjusting C_n^* in our choice of ρ_l we can ensure that

$$\sum_{k=1}^{\infty}|\delta\mathbf{v}^{r,\rho,k,l}|^n_{1,2}\leq\frac{1}{4}. \tag{321}$$

2.3 step 3: Construction of the function r and of a globally convergent iterative scheme for $\mathbf{v}^{r,\rho,l}$

We have proved the existence of local solutions for $\mathbf{v}^{r,\rho,l}=\mathbf{v}^{r,l}+\mathbf{r}^l$ for some fixed class of functions $\mathbf{r}^l$ and $\mathbf{v}^{r,\rho,l}$ with components in $C_b^{1,2}$ and such that the components are in the Hilbert space H^2 or in some Banach spaces $H^{s,p}$ in order to have convergence with some suitable regularity of the limit function. A 'suitable regularity' of the limit function is Hölder continuity which is uniform in time. This implies another simple reason for choosing the classical space: it allows us to have classical representations in terms of fundamental solutions not only for some approximating linear equations but also for the limit function once it is known. The classical representation in terms of fundamental solutions then allows us to get more regularity. Note that we used regularity of coefficients when we applied the adjoint equation in the first and second step of this proof. Furthermore the existence of classical solutions is useful because some forms of the maximum principle require the existence of classical solutions. These observations are useful for our construction of the function $\mathbf{r}$ which controls the global growth of the function $\mathbf{v}^{r,\rho}$ and is itself bounded (and has an integral magnitude which has linear growth with respect to time). The time step size at time step number l (measured in original time coordinates t) is ρ_l and we concluded that the local scheme converges with a time step size of order

$$\rho_l\leq\frac{1}{C_n^*\left(\left(C_{1,2}^{l-1}+C_r\right)+l\left(C_{1,2}^{l-1}+C_r\right)\right)4C_\Gamma^2}, \tag{322}$$

where $C_{1,2}^{l-1}$ is an upper bound for $\mathbf{v}^{r,\rho,l-1}$, where C_r a certain upper bound bound for $\mathbf{r}^l$ with respect to the $|.|_{1,2}$ norm. Indeed we determine the constant C_r in terms of the initial data function h_i in this section. Furthermore, at the end of this section we shall show that this choice of C_r is also an uniform upper bound for all $C_{1,2}^l$ if ρ_l is chosen as in (322). Concerning this constant C_n^* we note that it depends only on dimension and and may be computed explicitly. The constant C_Γ in (322) is a uniform bound of constants related to the fundamental solutions which appear at the kth substep of time step l (cf. step 1 of this proof). Step 1 and step 2 of this proof show: if the constants $C_{1,2}^{l-1}$ and C_r are given then C_Γ can be computed and local convergence is given along with the choice of ρ_l as in (322). (Note that we have shown in step 1 and step 2 of this proof that the other constants in (322) depend only on the dimension and the initial data $\mathbf{h}$ and, hence, are independent of the time step number l.)

We have to show that the numbers $C^l_{1,2},\ l \geq 0$ can be estimated in terms of a constant $C_{1,2}$ which is independent of l in order to get a global scheme. We choose a time step size of order $\rho_l \sim \frac{1}{l}$ in order to deal with the growth of the integral magnitudes related to the integral term of the Navier-Stokes equation in Leray projection form. Furthermore we have to show that the function $\mathbf{r}$ can be chosen together with a finite constant C_r independently of the time step number l. In this step 3 of the proof we determine the constant $C_{1,2}$ and the constant C_r which is a bound for the global function $\mathbf{r}$. In order to reduce the number of constants we shall determine $C_{1,2}$ such that

$$|r^l_i|_{1,2} \leq C_r = C_{1,2} \tag{323}$$

for all $l \geq 1$ and all $1 \leq i \leq n$. Since we have a step size with respect to original time coordinates of order $\rho_l \sim \frac{1}{l}$ we ensure that for the growth of the integral magnitude we have

$$\int_{\mathbb{R}^n} \sum_{j,k=1}^n \Big| \left(\frac{\partial r^l_k}{\partial x_j} \frac{\partial r^l_j}{\partial x_k} \right)(\tau, y) \Big| dy \leq C^*_n \left(C_{1,2} + lC_{1,2} \right), \tag{324}$$

i.e., we have linear growth of the integral magnitude with respect to the time step number l. This ensures that related integral magnitudes of the equation for $\mathbf{v}^{r,\rho,l}$ in Leray projection form have a linear bound, too. Note that the first derivative of the kernel K is bounded especially in case $n = 3$. In order to estimate the quantity (324) we may estimate

$$\int_{\mathbb{R}^n} \sum_{j,k,m,p=1}^n \Big| \frac{\partial r^l_k}{\partial x_j} \Big| \Big| \frac{\partial r^l_m}{\partial x_p} \Big| (\tau, y) dy \leq C^*_n \left(C_{1,2} + lC_{1,2} \right) \tag{325}$$

inductively. However, it is useful to reduce the estimate (327) to an L^2 estimate of the form

$$\int_{\mathbb{R}^n} \sum_{j,k,m,p=1}^n \left(\Big| \frac{\partial r^l_k}{\partial x_j} \Big|^2 + \Big| \frac{\partial r^l_m}{\partial x_p} \Big|^2 \right) (\tau, y) dy \leq C^*_n \left(C_{1,2} + lC_{1,2} \right). \tag{326}$$

Note that our local schemes of step 1 and step 2 of this proof shows that we need also estimates of derivatives of the integral terms, i.e., estimates of the form

$$\int_{\mathbb{R}^n} \sum_{j,k,m,p,q=1}^n \Big| \frac{\partial^2 r^l_k}{\partial x_j \partial x_q} \Big| \Big| \frac{\partial r^l_m}{\partial x_p} \Big| (\tau, y) dy \leq C^*_n \left(C_{1,2} + lC_{1,2} \right). \tag{327}$$

However, this can be treated similarly, i.e., all this reduces to an estimate in terms of a H^2 norm, i.e., to an estimate of form

$$|r^l_k(\tau, .)|_{H^2} \leq C^*_n \left(C_{1,2} + lC_{1,2} \right), \tag{328}$$

where we note that C_n^* is a generic constant.

Depending on the representation of the function r_i^l we may iterate the reduction of mixed products to sums of squares. Therefore we shall establish the estimate

$$\int_{\mathbb{R}^n}\left|\frac{\partial r_k^l}{\partial x_j}\right|^2(\tau,y)dy \leq C_{1,2}+lC_{1,2}, \tag{329}$$

and the estimate

$$\int_{\mathbb{R}^n}\left|\frac{\partial r_k^l}{\partial x_j}\right|^4(\tau,y)dy \leq C_{1,2}+lC_{1,2} \tag{330}$$

inductively when we have constructed the function r_i^l. This involves related estimates for functions $v_i^{r,\rho,l-1}$ and ϕ_i^l which occur in the construction of r_i^l. We shall show that $\mathbf{r}$ can be constructed via certain source term functions ϕ_i^l of linear parabolic equations such that (323) holds, and such that for all $l \geq 1$ we have

$$|\mathbf{v}^{r,\rho,l}|^n_{1,2} \leq C_{1,2} = C_r, \tag{331}$$

where C_r depends only on the dimension n and the initial data $\mathbf{h}$ of the Navier-Stokes Cauchy problem. We shall also show that

$$|\mathbf{v}^{r,\rho,l}|^n_{H^2} \leq C_n^*\left(C_{1,2}+lC_{1,2}\right), \tag{332}$$

and

$$|\mathbf{r}^{\rho,l}|^n_{H^2} \leq C_n^*\left(C_{1,2}+lC_{1,2}\right), \tag{333}$$

and that similar estimates are available with respect to other $H^{s,p}$-Banach spaces. This means that our method can be applied to general dimension n. This is outlined in the introduction of this section, and we shall fill in the details now and in the next step of this proof. In order to have an the equivalent problem (i.e., a problem equivalent to the original Navier-Stokes Cauchy problem for $\mathbf{v}$) for $\mathbf{v}^{r,\rho,l}$ we know that $\mathbf{r}$ has to be bounded (well, a bound linear with respect to the time step number l would be sufficient as well, but we show that we can construct a globally bounded function $\mathbf{r}$ such that the problem for $\mathbf{v}^{r,\rho}$ is equivalent to the problem for $\mathbf{v}$). The problem for $\mathbf{v}^{r,\rho}$ has the advantage that we can solve it step by step where we control the growth by the functions $\mathbf{r}^l$. Note that (331) implies that for the original equation we have the bound

$$|\mathbf{v}^{\rho,l}|^n_{1,2} \leq C_{1,2}+nC_r=(n+1)C_{1,2}. \tag{334}$$

The construction of the function $\mathbf{r}$ and of the global scheme for $\mathbf{v}^{r,\rho}$ is done inductively with respect to the time step number $l-1$. At each time step $l \geq 1$ assume that the functions $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ and $\mathbf{r}^{l-1}(l-1,.)$ have been computed. In case $l=1$ we set $\mathbf{v}^{r,\rho,l-1}(l-1,.)=\mathbf{h}$ and $\mathbf{r}^0\equiv 0$. We use these data and proceed at each time step l in three substeps as we outline next.

3i) At each time step $l \geq 1$ we construct first the functions ϕ^l_i which serve as 'consumption source terms' and are constructed such that the growth of the functions $\mathbf{v}^{\rho,r}$ and $\mathbf{r}^l$ is controlled on the domain $[l-1,l] \times \mathbb{R}^n$. Control of growth is both in the sense of supremum norms such as $|.|_0$ and $|.|_{1,2}$ and in the sense of integral norms such as $|.|_{L^2}$ and $|.|_{H^2}$. The functions ϕ^l_i, $1 \leq i \leq n$ are constructed as a sum of two functions, i.e.,

$$\phi^l_i = \phi^{v,l}_i + \phi^{r,l}_i, \tag{335}$$

where $\phi^{v,l}_i$ is mainly constructed in order to control the growth of $\mathbf{v}^{r,\rho,l}$ and $\phi^{r,l}_i$ is mainly constructed in order to control the growth of $\mathbf{r}^l$. Each of these functions is defined in terms of the data $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ and $\mathbf{r}^{l-1}(l-1,.)$ respectively. Concerning the growth in the sense of the supremum norm $|.|_0$ in the regions of the domain $[l-1,l] \times \mathbb{R}^n$ where the value of $v^{r,\rho,l-1}_i(l-1,.)$ is above a certain threshold $\frac{C}{2}$ we shall define $\phi^{v,l}_i = -1$ and in the regions of the domain $[l-1,l] \times \mathbb{R}^n$ where the value of $v^{r,\rho,l-1}_i(l-1,.)$ is below a certain threshold $-\frac{C}{2}$ we shall define $\phi^{v,l}_i = 1$. There are two possibilities here: for given $1 \leq i \leq n$ the modulus of the function r^l_i may be itself below or above the threshold $C/2$. We shall define a scheme such that in either case the growth of the function $v^{r,\rho,l}_i$ is controlled. Moreover, locally and for each i, if both function values $r^{l-1}_i(\tau,x)$ and $v^{r,\rho,l-1}_i(\tau,x)$ have opposite sign at some point (τ,x), then the construction of r^l_i and ϕ^l_i is such that the growth of all functions r^l_i, $v^{r,\rho,l}_i$, and $v^{\rho,l}_i$ is essentially controlled by the behavior of the function r^l_i if the latter function exceeds some level. In this way in special situations we control the growth of the function r^l_i and of the original Navier-Stokes solution $v^{\rho,l}_i$ and this leads to the growth control of the functions $v^{r,\rho,l}_i$ on $[l-1,l] \times \mathbb{R}^n$ (in τ-coordinates) is controlled. Similarly, in the regions of the domain $[l-1,l] \times \mathbb{R}^n$ where the value of the function $v^{r,\rho,l}_i$ is below a certain level, and the value of the function $r^{l-1}_i(l-1,.)$ is above a certain threshold $\frac{C}{2}$ we shall define $\phi^{r,l}_i = -1$. Furthermore in the regions of the domain $[l-1,l] \times \mathbb{R}^n$ where the value of the function $v^{r,\rho,l}_i$ is below a certain level, and the value of the function $r^{l-1}_i(l-1,.)$ is below a certain threshold $-\frac{C}{2}$ we shall define $\phi^{r,l}_i = 1$. The functions are defined via partitions of unity in order to have regular functions. The sum ϕ^l_i will appear on the right side of a linearized equation for $\mathbf{r}^l$ first. Furthermore, for each $1 \leq i \leq n$ we shall ensure that in the complementary regions where the modulus of both functions $r^{l-1}_i(l-1,.)$ or $v^{r,\rho,l-1}_i(l-1,.)$ is below $C/2$ the definition of the functions ϕ^l_i is extended such that the integral magnitudes of the functions r^l_i have a linear upper bound. Since products of functions can be pointwise estimated by sums of squares it is sufficient to have linear growth of H^2-estimates. The

estimation of the solution expression for r^l_i involves (first derivatives of) fundamental solutions of approximating linear equations integrated over time (one time step) and space. These integrals are absolutely bounded. The functions r^l_i are defined in the second substep which we sketch in advance next.

3ii) Once we have defined the functions ϕ^l_i we define the function $\mathbf{r}^l$ on the domain $[l-1,l]\times \mathbb{R}^n$ via linearized equations of the form

$$\begin{cases} r^l_{i,\tau}-\rho_l\nu\Delta r^l_i+\rho_l\sum_{j=1}^n r^{l-1}_j(l-1,.)\frac{\partial r^l_i}{\partial x_j}=\\ \\ +\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial r^{l-1}_j}{\partial x_k}\right)(l-1,y)dy\\ \\ -\rho_l\sum_{j=1}^n r^{l-1}_j\frac{\partial v^{r,\rho,l-1}_i}{\partial x_j}-\rho_l\sum_{j=1}^n v^{r,\rho,l-1}_j\frac{\partial r^{l-1}_i}{\partial x_j}\\ \\ +2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_k}\right)(l-1,y)dy\\ \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v^{r,\rho,l-1}_m}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_m}\right)(l-1,y)dy+\phi^l_i,\\ \\ \mathbf{r}^l(l-1,.)=\mathbf{r}^{l-1}(l-1,.). \end{cases} \tag{336}$$

Note that the functions on the right side with time index $l-1$ are defined on the domain $[l-2,l-1]\times\mathbb{R}^n$. Hence these functions are evaluated at $\tau=l-1$. Note that the function ϕ^l_i is defined on the domain $[l-1,l]\times\mathbb{R}^n$. However, we shall define ϕ^l_i as functions which depend only on the spatial variables, i.e., it they are constant with respect to time. We outlined the reasons for this in the introduction. Moreover, we assume that $\mathbf{r}^{l-1}(l-1,.)$ has been defined at the previous time step. At time step $l=1$ we have defined $\mathbf{r}^1\equiv 0$ Note that all functions on the right side of the first equation in (336) have a factor ρ_l except for the function ϕ^l_i. Choosing ρ_l small enough we ensure that the maximum value of each component of the function $r^l_i,\ 1\leq i\leq n$ of the function $\mathbf{r}^l$ is bounded in terms of the maximum of $r^l_i(l-1,.)=r^{l-1}_i(l-1,.),\ 1\leq i\leq n$ respectively for each i. Furthermore we shall see that we have a bound to the integral magnitude (324) where it is essential at time step l that ϕ^l_k is defined in substep 1 such that the L^p norm of

$$\int_{\mathbb{R}^n} r^{l-1}_k(l-1,y)\Gamma^l_r(\tau,x;s,y)dy+\int_0^\tau\int_{\mathbb{R}^n}\phi^l_k(s,y)\Gamma^l_r(\tau,x;s,y)dy, \tag{337}$$

is finite. Similarly for derivatives. The definition of ϕ_i^l in terms of the functions r_i^{l-1} and $v_i^{r,\rho,l-1}$ makes this possible.

3iii) Next we plug $\mathbf{v}^{r,\rho,l} = \mathbf{v}^{\rho,l} + \mathbf{r}^l$ into the Navier-Stokes equation on $[l-1,l]\times\mathbb{R}^n$. Then we get

$$\left\lbrace \begin{array}{ll} \frac{\partial v_i^{r,\rho,l}}{\partial \tau} - \rho_l\nu\sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,l}}{\partial x_j^2} + \rho_l\sum_{j=1}^n v_j^{r,\rho,l}\frac{\partial v_i^{r,\rho,l}}{\partial x_j} = \psi_i^l, \\ \\ \mathbf{v}^{r,\rho,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.), \end{array}\right. \tag{338}$$

where

$$\begin{array}{ll} \psi_i^l = r_{i,\tau}^l - \rho_l\nu\Delta r_i^l + \rho_l\sum_{j=1}^n r_j^l\frac{\partial r_i^l}{\partial x_j} \\ \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^l}{\partial x_j}\frac{\partial r_j^l}{\partial x_k}\right)(\tau,y)dy \\ \\ +\rho_l\sum_{j=1}^n r_j^l\frac{\partial v_i^{r,\rho,l}}{\partial x_j} + \rho_l\sum_{j=1}^n v_j^{r,\rho,l}\frac{\partial r_i^l}{\partial x_j} \\ \\ -2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^l}{\partial x_j}\frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy \\ \\ +\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l}}{\partial x_j}\frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy. \end{array} \tag{339}$$

In order to control the growth of $\mathbf{v}^{r,\rho,l}$ we consider the functional series $\mathbf{v}^{r,\rho,l} = \mathbf{v}^{r,\rho,0,l} + \sum_{k=1}^\infty \delta\mathbf{v}^{r,\rho,k,l}$ and consider a) the growth of $\mathbf{v}^{r,\rho,0,l}$, and b) the fact the sum of correction terms $\sum_{k=1}^\infty \delta\mathbf{v}^{r,\rho,k,l}$ is small for small ρ_l. The equation for

$$\left\lbrace \begin{array}{ll} \frac{\partial v_i^{r,\rho,0,l}}{\partial \tau} - \rho_l\nu\sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2} + \rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}(l-1,.)\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j} = \psi_i^{l,0} \\ \\ \mathbf{v}^{r,\rho,0,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.), \end{array}\right. \tag{340}$$

where

$$
\begin{array}{l}
\psi_i^{l,0}=r^l_{i,\tau}-\rho_l\nu\Delta r_i^l+\rho_l\sum_{j=1}^n r^l_j\frac{\partial r^l_i}{\partial x_j}\\
\\
-\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial r^l_j}{\partial x_k}\right)(\tau,y)dy\\
\\
+\rho_l\sum_{j=1}^n r^l_j\frac{\partial v^{r,\rho,l-1}_i}{\partial x_j}+\rho_l\sum_{j=1}^n v^{r,\rho,l-1}_j\frac{\partial r^l_i}{\partial x_j}\\
\\
-2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_k}\right)(\tau,y)dy\\
\\
+\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v^{r,\rho,l-1}_k}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_k}\right)(l-1,y)dy.
\end{array}
\tag{341}
$$

Then we analyze the growth of the functions $v_i^{r,\rho,0,l}$, $1\leq i\leq n$, where we estimate the $\psi^{l,0}_i$ using (336), and then estimate the difference $\psi^l_i-\psi^{l,0}_i$. Note that the estimate for the growth of $\mathbf{v}^{r,\rho,l}$ involves and estimate for the Navier-Stokes solution $\mathbf{v}^{\rho,l}$ and the control function $\mathbf{r}^l$ if the modulus of components of the latter function exceed a certain level in critical regions where the modulus functions $v_i^{r,\rho,l-1}$ exceed a certain critical level.

ad 3i) As we described in the introduction of this section the function ϕ^l_i is determined as a sum of two functions $\phi^{r,l}_i$ and $\phi^{v,l}_i$. Let us consider the first induction step $l=1$ first. At time step number $l=1$ we do not have a function $\mathbf{r}^{l-1}$ (all we have are the initial data $\mathbf{h}$). We may set $\mathbf{r}^0\equiv\mathbf{r}^1\equiv 0$. Hence there is no need to control the growth of $\mathbf{r}^1$ on the domain $[0,1]\times\mathbb{R}^n$ an we may set $\phi^{r,s,l}\equiv 0$. This means that we do not involve r^l_i in order to control the growth of the Navier-Stokes solution $v_i^{\rho,1}$ (in τ-coordinates) at the first time step $l=1$, i.e. on the domain $[0,1]\times\mathbb{R}^n$. Hence, ϕ^1_i is defined solely in terms of a function $\phi^{v,s,1}_i:\mathbb{R}^n\rightarrow\mathbb{R}^n$. Next we define $\phi^{v,s,1}_i$ to be a function which satisfies the following lemma with $B=D^0_{+,i}=\left\lbrace x|v^{r,\rho,l}_i(0,x)=h_i(x)\in\left[\frac{C}{2},C\right]\right\rbrace$, and
$A=D^{l-1}_{-,i}=\left\lbrace x|v^{r,\rho,l}_i(0,x)=h_i(x)\in\left[-C,-\frac{C}{2}\right]\right\rbrace$.

Lemma 2.19. *Let $A,B\subseteq\mathbb{R}^n$ be two closed disjoint sets. There is a function $f:\mathbb{R}^n\rightarrow\mathbb{R}$ such that*

1) $f\in C^2_b\left(\mathbb{R}^n\right)$, where

$$
|f|_{0,2}\leq C^+_n,
\tag{342}
$$

where C^+_n is a constant which depends only on the dimension n and

the distance of the sets A and B. (recall that we define

$$|f|_{0,2} = \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|\leq 2} |\partial_\alpha f(x)| \tag{343}$$

along with multivariate derivatives $\partial_\alpha = \frac{\partial^\alpha}{\partial x^\alpha}$ with multiindices $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $|\alpha| = \sum_i \alpha_i$, and where $\alpha_i \geq 0$ for all $1 \leq i \leq n$.)

2) *we have $\sup_{x\in\mathbb{R}^n} |f(x)| \leq 1$ by construction, and*

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ \\ -1 & \text{if } x \in B. \end{cases} \tag{344}$$

The lemma (2.19) can be proved using classical partitions of unity.

Proof. Let

$$\mu := \frac{\mathrm{dist}(A,B)}{2\sqrt{n}}, \tag{345}$$

where $\mathrm{dist}(A,B) := \sup_{x\in A, y\in B} |x-y|_2$ denotes the Euclidean distance. For integer tuples $p = (p_1, \cdots, p_n) \in \mathbb{Z}^n$ define a differentiable partition of unity

$$\phi_{p\mu}(x) = \Pi_{i=1}^n \psi\left(\frac{x_i}{\mu_i} - p_i\right) \tag{346}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is defined by

$$\psi(y) = \frac{h(y)}{H(y)} \tag{347}$$

along with

$$h(y) := \begin{cases} \exp\left(-\frac{1}{1-y^2}\right), \text{if } |y| \leq 1, \\ \\ 0 \ \ \text{else}, \end{cases} \tag{348}$$

and

$$H(y) = \sum_{m\in\mathbb{Z}} h(y-m). \tag{349}$$

Note that $\phi_{p\mu}$ has as its support cubes of side length μ of form

$$\mathrm{supp}(\phi_{p\mu}) = \{x \in \mathbb{R}^n | |x_i - p_i\mu| \leq \mu, \ 1 \leq i \leq n\}. \tag{350}$$

Furthermore, note that $(\phi_{p\mu})_{p\in\mathbb{Z}^n}$ is indeed a partition of unity, i.e.

$$\sum_{p\in\mathbb{Z}^n} \phi_{p\mu}(x) = 1 \text{ for all } \ x \in \mathbb{R}^n. \tag{351}$$

Next define the function $f : \mathbb{R}^n \to \mathbb{R}$ of the lemma via

$$f(x) = \sum_{p \in P_A} \phi_{p\mu}(x) - \sum_{p \in P_B} \phi_{p\mu}(x), \tag{352}$$

where

$$P_A := \{p \in \mathbb{Z}^n | \mathrm{supp}(\phi_{p\mu}) \cap A \neq \emptyset\}, \tag{353}$$

and

$$P_B := \{p \in \mathbb{Z}^n | \mathrm{supp}(\phi_{p\mu}) \cap B \neq \emptyset\}. \tag{354}$$

Computing the derivatives of f involves only finitely many terms (atmost 2^n either of the first sum indexed by P_A or of the second sum indexed by P_B. Especially we get

$$|f|_{0,2} \leq C_n^+ := 2^n C_\phi, \tag{355}$$

where $C_\phi = \max_{x \in \mathrm{supp}(\phi_{p\mu})} \sum_{|\alpha| \leq 2} \partial_\alpha \phi_{p\mu}(x)$. Note that the latter constant depends only on μ and n. □

Note that a fortiori the first part of the preceding lemma especially implies that

$$|f|_0 = \sup_{x \in \mathbb{R}^n} |f(x)| \leq C_n^+ \mathrm{dist}\,(A, B). \tag{356}$$

This implies that the distance of the sets A and B has a lower bound in terms of the norm $|f|_{0,1}$, i.e.

$$\mathrm{dist}\,(A, B) \geq \frac{1}{C_n^+} |f|_0, \tag{357}$$

where $C_n^+ > 0$ is some constant depending only on dimension n. Such a lower bound can be established independently of the time step l. It is clear that this fact is closely related to the boundedness of the functions $\mathbf{v}^{r,\rho}$ (and, hence, $\mathbf{v}^\rho$). It is also related to the linear bound of the integral magnitude in (12) and similar bounds for the functions $\mathbf{v}^{r,\rho}$ and $\mathbf{v}^\rho$. Clearly, we want to have a lower bound which is independent of the time step number l. We have now defined $\phi_i^1,\ 1 \leq i \leq n$ and $\mathbf{r}^1 \equiv 0$. The growth of the latter function is controlled by definition. Next we define the functions ϕ_i^l for $l \geq 2$. We assume that the functions $\mathbf{r}^{l-1}(l-1,.)$ and $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ are given. For each $1 \leq i \leq n$ the functions ϕ_i^l are constructed as a sum $\phi_i^l = \phi_i^{v,l} + \phi_i^{r,l}$ where both functions are determined in terms of the data $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ and $\mathbf{r}^{l-1}(l-1,.)$. These functions are obtained at the previous time step $l-1$. In order to construct the function $\phi_i^{v,l}$ we define

$$D_{+,i}^{v,l-1} = \left\{ x | v_i^{r,\rho,l-1}(l-1, x) \in \left[\frac{C}{2}, C\right], \right\} \tag{358}$$

and

$$D_{-,i}^{v,l-1} = \left\{ x | v_i^{r,\rho,l-1}(l-1, x) \in \left[-C, -\frac{C}{2}\right] \right\}. \tag{359}$$

Since $|v_i^{r,\rho,l-1}(l-1,.)|_{0,2}$ is bounded there is a distance

$$\mathrm{dist}\left(D_{+,i}^{v,l-1},D_{-,i}^{v,l-1}\right)\geq\delta \tag{360}$$

for some constant $\delta>0$. We shall show that this distance can be preserved at time step l. We apply lemma 2.19 with $A=D_{+,i}^{v,l-1}$ and $B=D_{-,i}^{v,l-1}$ for each $1\leq i\leq n$ and get the functions $\phi_i^{v,s,l}:\mathbb{R}^n\rightarrow\mathbb{R}$. We define

$$\phi_i^{v,l}:(l-1,l]\times\mathbb{R}^n\rightarrow\mathbb{R},\ \ \phi_i^{v,l}(\tau,x):=\phi_i^{v,s,l}(x), \tag{361}$$

and according to lemma 2.19 we have $\phi_i^{v,s,l}\in C_b^{1,2}$, where

$$\phi_i^{v,s,l}(x)=\begin{cases}-1 \ \text{ if } x\in D_{i,+}^{v,l-1},\\ \\ \alpha_i^l(x) \ \text{ else}\\ \\ 1 \ \text{ if } x\in D_{i,-}^{v,l-1},\end{cases} \tag{362}$$

for some function $\alpha\in C_b^{1,2}$ with $\alpha(x)\in[0,1]$ for all $x\in\mathbb{R}^n\setminus\left(D_{i,+}^{v,l-1}\cup D_{i,-}^{v,l-1}\right)$. This function $\alpha(x)$ is not arbitrary. Especially it is useful to ensure that $\alpha\in H^1$ on its domain $\mathbb{R}^n\setminus\left(D_{i,+}^{v,l-1}\cup D_{i,-}^{v,l-1}\right)$. Since $\phi_i^l=\phi_i^{v,l}+\phi_i^{r,l}$ appears as a source term of a linear equation for r_i^l we define

$$\alpha_i^l(x)=-\frac{2}{C}v_i^{r,\rho,l-1}(l-1,x). \tag{363}$$

where $x\in\mathbb{R}^n\setminus\left(D_{i,+}^{v,l-1}\cup D_{i,-}^{v,l-1}\right)=\mathbb{R}^n\setminus D_i^{v,l-1}$. On this domain this function is clearly in H^1 (by induction). Furthermore the function ϕ_i^l is globally Lipschitz, hence Hölder continuous. The functions $\phi_i^{r,l}$ are defined similarly. Recall that we defined subsets $D_{+,i}^{r,0,l-1}$ of $D_{+,i}^{r,l-1}$ and $D_{-,i}^{r,l-1}$ of $D_{-,i}^{r,0,l-1}$ which are mutually disjoint with the sets $D_{i,+}^{v,l-1},D_{i,-}^{v,l-1}$. We defined

$$D_{+,i}^{r,l-1}:=\left\{x|r_i^l(l-1,x)\in\left[\frac{C}{2},C\right]\ \right\}, \tag{364}$$

and

$$D_{-,i}^{r,l-1}:=\left\{x|r_i^l(l-1,x)\in\left[-C,-\tfrac{C}{2}\right]\ \right\}, \tag{365}$$

For our analysis it is important to consider certain subsets of the sets $D_{+,i}^{r,l-1}$ and $D_{-,i}^{r,l-1}$: the set

$$D_{+,i}^{r,0,l-1}:=\left\{x|r_i^l(l-1,x)\in\left[\frac{C}{2},C\right]\ \ \&\ \ v_i^{r,\rho,l-1}(l-1,x)\notin D_i^{v,l-1}\right\}, \tag{366}$$

and the set

$$D^{r,0,l-1}_{-,i} := \left\lbrace x|r^l_i(l-1,x)\in\left[-C,-\tfrac{C}{2}\right] \;\;\&\;\; v^{r,\rho,l-1}_i(l-1,x)\notin D^{v,l-1}_i\right\rbrace, \tag{367}$$

where we may define $\phi^{r,l}_i$ rather independently of $\phi^{v,l}_i$. On the complementary sets

$$D^{r,1,l-1}_{+,i} := D^{r,l-1}_{+,i}\setminus D^{r,0,l-1}_{+,i}, \tag{368}$$

and

$$D^{r,1,l-1}_{-,i} := D^{r,l-1}_{-,i}\setminus D^{r,0,l-1}_{-,i} \tag{369}$$

we ensure that we can control the growth of the functions $v^{r,\rho,l}_i$ and r^l_i simultaneously. How to do this is considered in detail in the third substep of this third substep. Again, since $|r^{l-1}_i(l-1,.)|_{0,2}$ is bounded there is a distance

$$\mathrm{dist}\left(D^{r,l-1}_{+,i},D^{r,l-1}_{-,i}\right)\geq\delta \tag{370}$$

for some constant $\delta>0$. Note that this constant δ is not a small constant but bounded from below uniformly as a consequence of the upper bound with respect to the $|.|_{0,2}$ norm of the functions r^{l-1}_i and $v^{r,\rho,l-1}_i$. We shall show that this distance can be preserved at time step l in the third substep below. Note that this implies that the constant δ in (370) can be chosen independently of the time step number l (cf. substep (iii) below). We apply lemma 2.19 with $A=D^{r,l-1}_{+,i}$ and $B=D^{r,l-1}_{-,i}$ for each $1\leq i\leq n$ and get the functions $\phi^{r,s,l}_i:\mathbb{R}^n\rightarrow\mathbb{R}$. We define

$$\phi^{r,l}_i:(l-1,l]\times\mathbb{R}^n\rightarrow\mathbb{R},\;\;\phi^{r,l}_i(\tau,x):=\phi^{v,s,l}_i(x), \tag{371}$$

and according to lemma 2.19 we have $\phi^{r,s,l}_i\in C^{1,2}_b$, where

$$\phi^{r,s,l}_i(x)=\begin{cases}-1 \;\text{ if } x\in D^{r,l-1}_{i,+},\\ \\ \beta(x)\;\text{ else}\\ \\ 1\;\text{ if } x\in D^{r,l-1}_{i,-},\end{cases} \tag{372}$$

for some function $\beta\in C^{1,2}_b$ with $\beta(x)\in[0,1]$ for all $x\in\mathbb{R}^n\setminus\left(D^{r,l-1}_{i,+}\cup D^{r,l-1}_{i,-}\right)$. Again the function β is not arbitrary. We define

$$\beta^l_i(x)=-\frac{2}{C}r^{l-1}_i(l-1,x). \tag{373}$$

on the domain $\mathbb{R}^n\setminus\left(D^{r,l-1}_{i,+}\cup D^{r,l-1}_{i,-}\right)$. Now we have constructed the functions $\phi^l_i=\phi^{v,l}_i+\phi^{r,l}_i$. This finishes substep i) of step 3, i.e., the defintion of the functions ϕ^l_i.

ad 3 ii): Next we construct the functions r^l_i for $1\leq i\leq n$. We want to define constants C^0_r, C^1_r, C_r which are independent of the time step number l and such that for all $l\geq 1$

$$|r^l_i|_0\leq C^0_r,\ |r^l_i|_{0,1}\leq C^1_r,\ |r^l_i|_{1,2}\leq C_r \tag{374}$$

holds, where the norms are defined as above with respect to the domain $D^\tau_l=(l-1,l]\times \mathbb{R}^n$. Furthermore we want to ensure linear growth of the intergal magnitude for the function $\mathbf{r}^l$ (see below). At the first time step $l=1$ we have defined $\mathbf{r}^1\equiv 0$. For $l\geq 2$ the functions r^l_i are defined by the Cauchy problem (336). Note that the right side is evaluated at $\tau=l-1$. We rewrite (336) in the form

$$\left\lbrace \begin{array}{ll} r^l_{i,\tau}-\rho_l\nu\Delta r^l_i+\rho_l\sum_{j=1}^n r^{l-1}_j(l-1,.)\frac{\partial r^l_i}{\partial x_j}=\\ \\ S^l_i(l-1,.)+\phi^l_i(l-1,.),\\ \\ \mathbf{r}^l(l-1,.)=\mathbf{r}^{l-1}(l-1,.). \end{array}\right. \tag{375}$$

where

$$\begin{array}{ll} S^l_i(l-1,x)=+\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial r^{l-1}_j}{\partial x_k}\right)(l-1,y)dy\\ \\ -\rho_l\sum_{j=1}^n r^{l-1}_j\frac{\partial v^{r,\rho,l-1}_i}{\partial x_j}-\rho_l\sum_{j=1}^n v^{r,\rho,l-1}_j\frac{\partial r^{l-1}_i}{\partial x_j}\\ \\ +2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_k}\right)(l-1,y)dy\\ \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v^{r,\rho,l-1}_m}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_m}\right)(l-1,y)dy. \end{array} \tag{376}$$

This is a linear parabolic equation for $\mathbf{r}^l$ which can be solved. Furthermore, the solution has a representation in terms of the fundamental solution Γ^l_r of the equation

$$r^l_{i,\tau}-\rho_l\nu\Delta r^l_i+\rho_l\sum_{j=1}^n r^{l-1}_j(l-1,.)\frac{\partial r^l_i}{\partial x_j}=0, \tag{377}$$

which allows us to analyze the growth of $\mathbf{r}^l$. We have

$$\begin{array}{ll} r^l_i(\tau,x)=\int_{\mathbb{R}^n}r^{l-1}_i(l-1,y)\Gamma^l_r(\tau,x;0,y)dy\\ \\ +\int_{l-1}^\tau\int_{\mathbb{R}^n}S^l_i(l-1,y)\Gamma^l_r(\tau,x;s,y)dsdy\\ \\ +\int_{l-1}^\tau\int_{\mathbb{R}^n}\phi^l_i(s,y)\Gamma^l_r(\tau,x;s,y)dsdy, \end{array} \tag{378}$$

We shall exploit the following ideas related to the representation (378): concerning the the relation of the second and the third term on the right side of (378) we note that all terms of the function S^l have the factor ρ_l while the third term ϕ^l_i does not have this factor. If the factor $\rho_l > 0$ is small, then the diffusive effect of the fundamental solution Γ^l_r is small in the sense that the values of the second and third term on the right side of (378) depend largely on the source terms $S^l_i(l-1,y)$ and $\phi^l_i(s,y)$ (we shall make this remark precise in a moment). This means that for ρ_l small enough for each $1 \leq i \leq n$ the term in (378) involving ϕ^l_i dominates the term involving S^l_i. Furthermore a bound for the first term of the right side in (378) can be obtained using the maximum principle. Considering derivatives and shifting of derivatives using the adjoint of the fundamental solution as we have done in the first step of this proof leads to an estimate of the functions r^l_i in the $|.|_{1,2}$ norm. Furthermore, we have the estimates

$$\begin{array}{ll}
|r^l_i(\tau,.)|_{H^1} \leq \Big| \sum_{j=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} r^{l-1}_i(l-1,y)\left(|\Gamma^l_r(\tau,x;0,y)dy| + |\Gamma^l_{r,j}(\tau,x;0,y)dy|\right)\\
+\int_{l-1}^{\tau}\int_{\mathbb{R}^n} \phi^l_i(s,y)\left(|\Gamma^l_r(\tau,x;s,y)dy| + |\Gamma^l_{r,j}(\tau,x;s,y)dy|\right)dsdy\Big|\\
\leq C^*_n \Big|r^{l-1}_k\Big|_{H^1} + C^*\Big|v^{r,\rho,l-1}_k\Big|_{H^1}
\end{array}\tag{379}$$

and

$$\begin{array}{ll}
|r^l_i(\tau,.)|_{H^2} \leq \Big| \sum_{j,m=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} r^{l-1}_{i,m}(l-1,y)\left(|\Gamma^{l,*}_r(\tau,x;0,y)dy| + |\Gamma^{l,*}_{r,j}(\tau,x;0,y)dy|\right)\\
+\int_{l-1}^{\tau}\int_{\mathbb{R}^n} \phi^l_{i,m}(s,y)\left(|\Gamma^{l,*}_r(\tau,x;s,y)dy| + |\Gamma^{l,*}_{r,j}(\tau,x;s,y)dy|\right)dsdy\Big|\\
\leq C^*_n \Big|r^{l-1}_k\Big|_{H^2} + C^*\Big|v^{r,\rho,l-1}_k\Big|_{H^2}
\end{array}\tag{380}$$

with our generic constant $C^*_n > 0$ independent of the time step number l. In this step we use the definition of ϕ^l_i and the adjoint as in step 1. The latter fact then ensures that we have a linear growth of the intergal magnitude for the functions r^l_i, i.e. the magnitude

$$\int_{\mathbb{R}^n}\sum_{j,k=1}^n \left|\frac{\partial r^l_k}{\partial x_j}(\tau,y)\right|\left|\frac{\partial r^l_j}{\partial x_k}(\tau,y)\right|dy. \tag{381}$$

is bounded by $C_{1,2} + lC_{1,2}$ for $\tau \in [l-1,l]$. Indeed we shall establish this linear bound for an upper bound of (381), i.e., we shall find that

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n \left|\frac{\partial r^l_k}{\partial x_j}(\tau,y)\right|\left|\frac{\partial r^l_m}{\partial x_p}(\tau,y)\right|dy \leq C_{1,2} + lC_{1,2}. \tag{382}$$

Since we estimate the functions $v_i^{r,\rho,l}(\tau,.)$ and $r_i^l(\tau,.)$ with respect to the norm $|.|_{0,2}$ (this means spatial derivatives up to second order) we construct similar bounds for the first derivatives of (382). Again, since products of functions can be pointwise estimated by sums of squares estimates the functions $v_i^{r,\rho,l}(\tau,.)$ and $r_i^l(\tau,.)$ with respect to the H^2 norm are essential. Note that the choice of $\rho_l \sim \frac{1}{l}$ compensates for this linear growth in order to esure convergence of the local scheme with respect to intergal norms of step 2 of this proof. Next we have to determine the upper bounds C_r and $C_{1,2}$ of the functions $\mathbf{r}$ and $\mathbf{v}^{r,\rho}$. We define the constants $C_{1,2} = C_r$ just in terms of the initial data function $\mathbf{h}$ (dependence on the viscosity νmay be put into the step size). This determines ρ_l, and we check that r^l is bounded independently of the time step number l and that the integral magnitude of $\mathbf{r}^l$ is bounded, too. We define

$$\begin{array}{l} C_r = C_{1,2} = 2 + 2|\mathbf{h}|^n_{1,2} + \int_{\mathbb{R}^n} \sum_{j,k,l,m=1}^n \left|\frac{\partial h_k}{\partial x_j}(y)\right|\left|\frac{\partial h_l}{\partial x_m}(y)\right|dy \\ + \int_{\mathbb{R}^n} \sum_{j,k,l,m,p=1}^n \left|\frac{\partial^2 h_k}{\partial x_j \partial x_p}(y)\right|\left|\frac{\partial h_l}{\partial x_m}(y)\right|dy. \end{array} \tag{383}$$

The last term in the definition (383) is due to the fact that we do the estimates for local $|.|_{1,2}$-norms, i.e., on the domains $(l-1,l]\times\mathbb{R}^n$ and global H^2 norms for the functions $v_i^{r,\rho,l}(\tau,.)$ and $r_i^l(\tau,.)$. Next we prove that the constants C_r^0, C_r^1 and C_r are bounds for the functions r_i^l with respect to the norms $|.|_0$, $|.|_{0,1}$, and $|.|_{1,2}$ which are preserved at the lth time step. Assume inductively that

$$|r_i^{l-1}|_0 \leq C_r^0,\ |r_i^{l-1}|_{0,1} \leq C_r^1,\ |r_i^{l-1}|_{1,2} \leq C_r. \tag{384}$$

We have not defined C_r^0 and C_r^1 yet but it turns out that the constant C_r is essential. Hence we just define

$$C_r^0 := C_r, \text{ and } C_r^1 := C_r. \tag{385}$$

Clearly a closer analysis leads to some bounds $C_r^0, C_r^1 \leq C_r$, but this is not essential. However we define r_i^l, equation for r_i^l includes an the integral term of the form which we know from the Navier-Stokes equation in its Leray projection form. At time step l this term may be given in terms of the function r_i^{l-1} of the previous time step. Hence, it may seem natural to assume inductively that

$$\int_{\mathbb{R}^n} \sum_{j,k,m,p=1}^n \Big|\frac{\partial r_k^{l-1}}{\partial x_j}(l-1,y)\Big|\Big|\frac{\partial r_m^{l-1}}{\partial x_p}(l-1,y)\Big|dy \leq C_r + (l-1)C_r \tag{386}$$

Preservation of these bounds at the lth time step means that

$$|r_i^l|_0 \leq C_r^0,\ |r_i^l|_{0,1} \leq C_r^1,\ |r_i^l|_{1,2} \leq C_r, \tag{387}$$

and that

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n \Big|\frac{\partial r^l_k}{\partial x_j}(\tau,y)\Big|\Big|\frac{\partial r^l_m}{\partial x_p}(\tau,y)\Big|dy\leq C_r+lC_r \tag{388}$$

for all $\tau\in [l-1,l]$. Since we have not estimated $\mathbf{v}^{r,\rho,l-1}$ yet we can prove this only in terms of estimates of the latter function which are inductively assumed. Our local method described in the first and second substep of this proof indicates that we should require a similar requirement for the firs spatial derivative, i.e., that we should require inductively that

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^n \Big|\frac{\partial^2 r^{l-1}_k}{\partial x_j\partial x_q}(l-1,y)\Big|\Big|\frac{\partial r^{l-1}_m}{\partial x_p}(l-1,y)\Big|dy\leq C_r+(l-1)C_r \tag{389}$$

Nite that by definition the functions r^l_i depend on the source terms ϕ^l_i of their defining equations, and these source terms depend on $v^{r,\rho,l-1}_i(l-1,.)$ and r^{l-1}_i in turn. Hence, we assume inductively that

$$|v^{r,\rho,l-1}_i|_0\leq C^{l-1}_0,\ |v^{r,\rho,l-1}_i|_{0,1}\leq C^{l-1}_1,\ \ |v^{r,\rho,l-1}_i|_{0,2}\leq C^{l-1}_{0,2},\ |v^{r,\rho,l-1}_i|_{1,2}\leq C^{l-1}_{1,2} \tag{390}$$

for some finite constants $C^{l-1}_0,\ C^{l-1}_1,\ C^{l-1}_{1,2}$. In the end we shall have

$$C^{l-1}_{1,2}\leq C_{1,2}:=C_r \tag{391}$$

and we may set $C^{l-1}_0:=C_{1,2}$, and $C^{l-1}_1=C^{l-1}_{1,2}$, and $C^{l-1}_{0,2}:=C^{l-1}_{1,2}$, but we shall show this later in substep (iii) below. Hence we keep the index l for this time. Furthermore, we assume inductively that

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n \Big|\frac{\partial v^{r,\rho,l-1}_k}{\partial x_j}(l-1,y)\Big|\Big|\frac{\partial v^{r,\rho,l-1}_m}{\partial x_p}(l-1,y)\Big|dy\leq C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}, \tag{392}$$

and substep (iii) below shows that this linear growth and that of the

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^n \Big|\frac{\partial^2 v^{r,\rho,l-1}_k}{\partial x_j\partial x_q}(l-1,y)\Big|\Big|\frac{\partial v^{r,\rho,l-1}_m}{\partial x_p}(l-1,y)\Big|dy\leq C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}, \tag{393}$$

is preserved as well. Well all the inductive assumptions on integral terms involving r^{l-1}_i and $v^{r,\rho,l-1}_i$ can be expressed by one condition which is

$$\begin{array}{ll}\big|v^{r,\rho,l-1}_i(l-1,.)\big|_{H_2}+\big|r^{l-1}_i(l-1,.)\big|_{H_2}\\ \leq C^*_n\left(C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}+C_r+(l-1)C_r\right).\end{array} \tag{394}$$

Next we do the induction step. Note that the first term on the right side of (378) can estimated using the maximum principle for a corresponding

parabolic equation without source term. Next we use the fact that the fundamental solution Γ^l_r is positive (nonnegative would suffice at this point). Indeed, since Γ^l_r solves a uniform parabolic problem with constant second order and bounded first order coefficients r_i^{l-1} which are Hölder continuous with respect to the spatial variables uniformly in time, we have

$$\Gamma^l_r > 0 \tag{395}$$

(cf. Thm. 11 of chapter 2 of [3] for a proof). Next we estimate the functions S^l_i. We consider the right side of equation (376) term by term. The essential estimate is with respect to the supremum norm $|.|_0$ which we do first. We have

$$\left|\rho_l \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^{l-1}}{\partial x_j}\frac{\partial r_j^{l-1}}{\partial x_k}\right)(l-1,y)dy\right|_0 \leq \rho_l C_K\left(C_r+lC_r\right), \tag{396}$$

where we use (389). Next we have for the second and third term on the right side of (376) the estimate

$$\left|\rho_l\sum_{j=1}^n r_j^{l-1}\frac{\partial v_i^{r,\rho,l-1}}{\partial x_j}\right| + \left|\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}\frac{\partial r_i^{l-1}}{\partial x_j}\right|_0 \leq 2n\rho_l C_1^{l-1}C_r \tag{397}$$

For the fourth term on the right side of (376) we have the estimate

$$\begin{array}{ll}+\left|2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^{l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(l-1,y)dy\right|_0\\ \\ \leq \rho_l C_K\left(C_r+lC_r\right)+\rho_l C_K\left(C_1^{l-1}+lC_1^{l-1}\right),\end{array} \tag{398}$$

where we use

$$\begin{array}{ll}\left(\frac{\partial r_k^{l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(l-1,y)\\ \\ \leq \frac{1}{2}\left(\frac{\partial r_k^{l-1}}{\partial x_j}\right)^2(l-1,y)+\left(\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)^2(l-1,y).\end{array} \tag{399}$$

Finally, we have the estimate

$$\begin{array}{ll}\left|\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v_m^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_m}\right)(l-1,y)dy\right|\\ \\ \leq \rho_l C_K\left(C_1^{l-1}+lC_1^{l-1}\right).\end{array} \tag{400}$$

Summing up the four estimates (396), (397), (398), (400) we get

$$\begin{array}{ll}|S^l_i(l-1,.)|_0\\ \\ \leq \rho_l C_K\left(2C_1^{l-1}+l2C_1^{l-1}+2C_r+l2C_r\right)+2n\rho_l C_1^{l-1}C_r,\end{array} \tag{401}$$

Considering first derivatives we get

$$
\begin{aligned}
&|S^l_i(l-1,.)|_{0,1}\\
&\leq \rho_l 2C_K\left(2C^{l-1}_{0,2}+l2C^{l-1}_{0,2}+2C_r+l2C_r\right)+4n\rho_l C^{l-1}_{0,2}C_r,
\end{aligned}
\tag{402}
$$

where we note the generosity that we used of the constant C_r in the estimate (401). Hence some choice

$$
\rho_l\leq \frac{1}{4\left(2C_K\left(2C^{l-1}_{0,2}+l2C^{l-1}_{0,2}+2C_r+l2C_r\right)+4nC^{l-1}_{0,2}C_r\right)}.
\tag{403}
$$

leads to

$$
|S^l_i(l-1,.)|_{0,1}\leq \frac{1}{4}.
\tag{404}
$$

Note that (403) can be put into the form (322) above, i.e. ino the form

$$
\rho_l\leq \frac{1}{C^*_n\left(\left(C^{l-1}_{1,2}+C_r\right)+l\left(C^{l-1}_{1,2}+C_r\right)\right)4C^2_{\Gamma}},
\tag{405}
$$

where $C_{\Gamma}\geq 1$ without loss of generality and and the generic constant C^*_n absobs all the constants which depend on dimension n. Note that the constant ν is implicitly in C_{Γ}. Next we apply

Lemma 2.20. *Assume that ρ_l is sufficiently small. Then for $1\leq i\leq n$, and for $(\tau,x)\in D^{l-1}_{+,i}$ we have*

$$
\int_{l-1}^{\tau}\phi^l_i(s,y)\Gamma^l_r(\tau,x;s,y)dsdy\leq -\frac{3}{4}\left(\tau-(l-1)\right),
\tag{406}
$$

and for $(\tau,x)\in[l-1,l]\times\mathbb{R}^n$ we have

$$
\int_{l-1}^{\tau}\phi^l_i(s,y)\Gamma^l_r(\tau,x;s,y)dsdy\geq \frac{3}{4}\left(\tau-(l-1)\right).
\tag{407}
$$

Proof. Note that for $(\tau,x)\in D^{l-1}_{+,i}$ means that $\phi^l_i(s,y)$ You check first that this follows for the heat kernel instead of Γ^l_r. Then you use the Levy expansion representation for Γ^l_r and show that for ρ_l it is a perturbation of of the heat kernel. □

Similarly we have

Lemma 2.21. *Assume that ρ_l is sufficiently small. Then for $1\leq i\leq n$, and for $(\tau,x)\in D^{l-1}_{+,i}$ we have*

$$
|\int_{l-1}^{\tau}S^l_i(s,y)\Gamma^l_r(\tau,x;s,y)dsdy|\leq \frac{1}{2}\left(\tau-(l-1)\right).
\tag{408}
$$

Next we assume that ρ_l is small enough. Note that we assume inductively that the coefficients r_i^{l-1} are bounded by C_r (even with repsect to the $|.|_{1,2}$ norm on $(l-1,l]\times\mathbb{R}^n$- hence saying that ρ_l is small enough means that $\rho_l C_r$ is small enough such that the proof of the preceding lemmas can be applied. Next for $(\tau,x)\in D^{l-1}_{+,i}$ we have

$$\begin{array}{ll}|r_i^l(\tau,x)|\leq \sup_{(\tau,x)\in[l-1,l]\times\mathbb{R}^n}\Big|C_r^0+\\ \int_{l-1}^{\tau}\int_{\mathbb{R}^n}\left(S(l-1,y)+\phi_i^l(s,y)\right)\Gamma_r^l(\tau,x;s,y)dsdy\Big|\leq C_r^0.\end{array}\tag{409}$$

Similarly, if $(\tau,x)\in D^{l-1}_{-,i}$ we have

$$\begin{array}{ll}r_i^l(\tau,x)\geq \int_{\mathbb{R}^n} r_i^{l-1}(l-1,y)\Gamma_r^l(\tau,x;0,y)dy+\\ \int_{l-1}^{\tau}\int_{\mathbb{R}^n}\left(S(l-1,y)+\phi_i^l(s,y)\right)\Gamma_r^l(\tau,x;s,y)dsdy\\ \geq -C_r^0.\end{array}\tag{410}$$

Furthermore, for $(\tau,x)\notin D^{l-1}_{-,i}\cup D^{l-1}_{+,i}$ we have

$$|r_i^{l-1}(\tau,x)|\leq\frac{C_r^0}{2},\tag{411}$$

and, hence, (since $C_r^0\geq 2$) we have

$$\begin{array}{ll}|r_i^l(\tau,x)|\leq \sup_{(\tau,x)\in[l-1,l]\times\mathbb{R}^n}\Big|\frac{C_r^0}{2}+\\ \int_{l-1}^{\tau}\int_{\mathbb{R}^n}\left(S(l-1,y)+\phi_i^l(s,y)\right)\Gamma_r^l(\tau,x;s,y)dsdy\Big|\leq C_r^0.\end{array}\tag{412}$$

Hence, we get for $l\geq 2$

$$\sup_{x\in\mathbb{R}^n}|r_i^l(\tau,x)|\leq\sup_{x\in\mathbb{R}^n}|r_i^l(l-1,x)|\leq C_r^0\tag{413}$$

for all $\tau\in[l-1,l]$. Next we look at derivatives. Note that the functions ϕ_i^l are designed in order to control the growth of the functions r_i^l themselves, not the derivatives of that functions. Recall that the functions $r_i^{l-1}(l-1,.)=r^l(l-1,.)$ are the first order coefficients of the linear parabolic equations which determine the fundamental solutions Γ_r^l. Hence, in order to have classical solutions r_i^l it is sufficient to control the $|.|_{0,1}$ of the functions $r_i^{l-1}(l-1,.)$ for all $l\geq 1$. Nevertheless, we can control the growth of the derivatives up to second order since second order derivative estimates can be reduced to first order derivative estimates and the supremum norm of the derivatives of $r_i^l(l-1,.)$ can be estimated in terms of the supremum norm of r_i^l itself. For

the first spatial derivatives we use properties of the fundamental solution. For the second spatial derivatives we use the adjoint in addition. Recall that we use the constant C^*_n -which depends on n essentially- generically. We show

Lemma 2.22. *There is a constant $C^*_n > 0$ such that the estimates*

$$\sup_{x\in\mathbb{R}^n} |r^l_{i,j}(l,x)| \leq C^*_n \sup_{x\in\mathbb{R}^n} |r^l_i(l-1,x)|, \tag{414}$$

and

$$\sup_{x\in\mathbb{R}^n} |r^l_{i,j,k}(l,x)| \leq C^*_n \sup_{x\in\mathbb{R}^n} |r^l_i(l-1,x)| \tag{415}$$

hold. Here, recall the notation

$$r^l_{i,j}(\tau,x) := \frac{\partial}{\partial x_j} r^l_i(\tau,x),\ \ r^l_{i,j,k}(\tau,x) := \frac{\partial^2}{\partial x_j \partial x_k} r^l_i(\tau,x) \tag{416}$$

Proof. For the first derivatives we have the expression

$$\begin{array}{ll}\frac{\partial}{\partial x_k} r^l_i(\tau,x) = \int_{\mathbb{R}^n} r^{l-1}_i(l-1,y)\Gamma^l_{r,k}(\tau,x;0,y)dy\\ +\int_{l-1}^{\tau}\int_{\mathbb{R}^n}\left(S^l_i(l-1,y)+\phi^l_i(s,y)\right)\Gamma^l_{r,k}(\tau,x;s,y)dsdy\end{array} \tag{417}$$

Next we consider the fundamental solution Γ^l_r. It has the Levy expansion:

$$\Gamma^l_r(\tau,x;s,y) := N^l(\tau,x;s,y) + \int_s^{\tau}\int_{\mathbb{R}^n} N^l(\tau,x;\sigma,\xi)\phi_r(\sigma,\xi;s,y)d\sigma d\xi, \tag{418}$$

where

$$N^l(\tau,x;s,y) = \frac{1}{\sqrt{4\pi\rho_l\nu(\tau-s)}^n}\exp\left(-\frac{|x-y|^2}{4\rho_l\nu(\tau-s)}\right), \tag{419}$$

and ϕ_r is a recursively defined function which is Hölder continuous in x, i.e.,

$$\phi_r(\tau,x;s,y) = \sum_{m=1}^{\infty}(L^r_l N^l)_m(\tau,x;s,y), \tag{420}$$

along with the recursion

$$\begin{array}{ll}(L^r_l N^l)_1(\tau,x;s,y) = L^r_l N^l(\tau,x;s,y)\\ = \frac{\partial N^l}{\partial \tau} - \rho_l\nu\Delta N^l + \rho_l\sum_{j=1}^n r^{l-1}_j\frac{\partial N^l}{\partial x_j}\\ = \rho_l\sum_{j=1}^n r^{l-1}_j\frac{\partial N^l}{\partial x_j},\\ (L^r N^l)_{m+1}(\tau,x) := \int_s^t\int_{\Omega}\left(L^r N^l(\tau,x;\sigma,\xi)\right)_m L^r N^l(\sigma,\xi;s,y)d\sigma d\xi.\end{array} \tag{421}$$

The lemma may be proved for N_l instead of Γ^l_r in the representation (417) first and then the proof may be extended to the correction terms involving $\int_s^\tau \int_{\mathbb{R}^n} N^l(\tau,x;\sigma,\xi)\phi_r(\sigma,\xi;s,y)d\sigma d\xi$ using the classical Levy expansion estimates. The proof may be obtained in sub-timesteps, i.e. the first the norm

$$\sup_{x\in\mathbb{R}^n} |r^l_{i,j}(l-\frac{1}{2},x)| \leq C^*_n \sup_{x\in\mathbb{R}^n} |r^l_i(l,x)| \leq C^*_n \sup_{x\in\mathbb{R}^n} |r^l_i(l,x)|. \tag{422}$$

Then we may use

$$\begin{array}{ll} N^l_{,k}(\tau,x;s,y) = \frac{(x-y)_k}{4\rho_l\nu(\tau-s)}\frac{1}{\sqrt{4\pi\rho_l\nu(\tau-s)}^n}\exp\left(-\frac{|x-y|^2}{4\rho_l\nu(\tau-s)}\right) \\ = \frac{(x-y)_k}{4\rho_l\nu(\tau-s)}\exp\left(-\frac{|x-y|^2}{8\rho_l\nu(\tau-s)}\right)\frac{\sqrt{2}^n}{\sqrt{8\pi\rho_l\nu(\tau-s)}^n}\exp\left(-\frac{|x-y|^2}{8\rho_l\nu(\tau-s)}\right), \end{array} \tag{423}$$

and

$$\frac{(x-y)^2_k}{4\rho_l\nu(\tau-s)}\exp\left(-\frac{|x-y|^2}{16\rho_l\nu(\tau-s)}\right)\frac{1}{(x-y)_k}\exp\left(-\frac{|x-y|^2}{16\rho_l\nu(\tau-s)}\right) \leq C \tag{424}$$

for some $C>0$. This leads to the first statement of the lemma. Finally we use the argument above that

$$\sup_{x\in\mathbb{R}^n} |r^l_i(l,x)| \leq \sup_{x\in\mathbb{R}^n} |r^l_i(l-1,x)|. \tag{425}$$

For the second derivatives we use partial integration and the adjoint

$$\begin{array}{ll} \frac{\partial^2}{\partial x_k \partial x_m} r^l_{i,k}(\tau,x) = \int_{\mathbb{R}^n} r^{l-1}_{i,k}(l-1,y)\Gamma^{*,l}_{r,m}(\tau,x;0,y)dy \\ + \int_{l-1}^\tau \int_{\mathbb{R}^n} S^l_{i,k}(l-1,y)\Gamma^{*,l}_{r,m}(\tau,x;s,y)dsdy \\ + \int_{l-1}^\tau \int_{\mathbb{R}^n} \phi^l_{i,k}(s,y)\Gamma^{*,l}_{r,m}(\tau,x;s,y)dsdy. \end{array} \tag{426}$$

It follows that the estimate can be reduced to the estimate for first derivatives. Since C^*_n is generic we may replace $(C^*_n)^2$ by C^*_n.

□

Next we observe that we have an upper bound for the integral magnitudes for $\mathbf{r}^l$ and its derivatives is preserved, i.e., we want to show that

$$\int_{\mathbb{R}^n} \sum_{j,k,m,p=1}^n \Big|\frac{\partial r^l_k}{\partial x_j}\Big|\Big|\frac{\partial r^l_m}{\partial x_p}\Big|(t,y)dy \leq C_r + lC_r \tag{427}$$

and

$$\int_{\mathbb{R}^n} \sum_{j,k,m,p,q=1}^n \Big|\frac{\partial^2 r^l_k}{\partial x_j \partial x_q}\Big|\frac{\partial r^l_m}{\partial x_p}\Big|(t,y)dy \leq C_r + lC_r \tag{428}$$

for $1\leq k\leq n$, where we have assumed this upper bound for $l-1$ inductively. The estimates (429) and (430), which are of a form which is closely related to the integral term of the Navier-Stokes equation in Leray projection form - follow from the H^1- and H^2-type estimates

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^{n}\left(\Big|\frac{\partial r^l_k}{\partial x_j}\Big|^2+\Big|\frac{\partial r^l_m}{\partial x_p}\Big|^2\right)(t,y)dy\leq C_r+lC_r \tag{429}$$

and

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^{n}\left(\Big|\frac{\partial^2 r^l_k}{\partial x_j\partial x_q}\Big|^2+\Big|\frac{\partial r^l_m}{\partial x_p}\Big|^2\right)(\tau,y)dy\leq C_r+lC_r \tag{430}$$

Therefore we assume this inductively. Similar for the function $v^{r,\rho,l}_i$. We assume inductively that

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^{n}\Big|\frac{\partial v^{r,\rho,l-1}_k}{\partial x_j}\Big|\Big|\frac{\partial v^{r,\rho,l-1}_m}{\partial x_p}\Big|(\tau,y)dy\leq C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}. \tag{431}$$

and

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^{n}\Big|\frac{\partial^2 v^{l-1}_k}{\partial x_j\partial x_q}\Big|\Big|\frac{\partial v^{l-1}_m}{\partial x_p}\Big|(\tau,y)dy\leq C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}. \tag{432}$$

Again this follows from estimates of type

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^{n}\left(\Big|\frac{\partial v^{r,\rho,l-1}_k}{\partial x_j}\Big|^2+\Big|\frac{\partial v^{r,\rho,l-1}_m}{\partial x_p}\Big|^2\right)(\tau,y)dy\leq C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}, \tag{433}$$

and

$$\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^{n}\left(\Big|\frac{\partial^2 v^{r,\rho,l-1,}_k}{\partial x_j\partial x_q}\Big|^2+\Big|\frac{\partial v^{r,\rho,l-1}_m}{\partial x_p}\Big|^2\right)(\tau,y)dy\leq C^{l-1}_{1,2}+(l-1)C^{l-1}_{1,2}. \tag{434}$$

In the next substep we shall see that there is an upper bound $C_{1,2}$ for all $C^l_{1,2}$ which is independent of the time step number l.

In order to find these upper bounds (429) and (430) we go back to the representation of r^l_i in (378). The first term in (378) is an integral of the initial data r^{l-1}_i times the fundamntal solution Γ^l_r. The second order coefficients $\rho_l\nu$ clearly have a uniform upper bound and the first order coefficients r^{l-1}_i (which are also data from the previous time step) have a uniform upper bound by inductive assumption. We also know that the first spatial derivative of the fundamental solution is locally and globally integrable for fixed $x\in\mathbb{R}^n$ (especially, the integral with respect to y is in L^∞). We may use

the adjoint for the first and second derivatives in order to estimate the first term on the right side of the representation in (378) in terms of expressions of the for (429) and (430). Involvement of the fundamental solution Γ^l_r leads to a factor $C^*_n = 2C^2_\Gamma$, where we may use our generic constant C^*_n since C_Γ depends on dimension n essentially and is defined similar as in the first step of this proof, i.e.,

$$\sup_{(\tau,x)\in D^\tau_l} \int_{l-1}^{\tau} \int_{\mathbb{R}^n} \sup_{|\mathbf{r}|^n_{1,2}\in C_r} \left(|\Gamma^l_r| + |\Gamma^l_{r,i}|\right) dyds =: C_\Gamma. \tag{435}$$

Again local integrability and Gaussian-type a priori estimates lead to the conclusion that C_Γ is finite. Note that we can estimate products by sums of squares in order to estimate mixed terms. There is one difference when we estimate supremum norms and integral norms of r^l_i. Note that we have defined $\phi^l_i = \phi^{r,l}_i + \phi^{v,l}_i$ in terms of the functions r^{l-1}_i and $v^{r,\rho,l-1}_i$ such that the H^1-and H^2-norms of ϕ^l_i are bounded by the sum of the H^1-and H^2-norms of r^{l-1}_i and $v^{r,\rho,l-1}_i$. Moreover, by the very definition of ϕ^l_i this holds also for the function (note that ρ_l is small)

$$\begin{array}{l} R^l_i(\tau,x) = \int_{\mathbb{R}^n} r^{l-1}_i(l-1,y)\Gamma^l_r(\tau,x;0,y)dy \\ \\ +\int_{l-1}^{\tau}\int_{\mathbb{R}^n} \phi^l_i(s,y)\Gamma^l_r(\tau,x;s,y)dsdy. \end{array} \tag{436}$$

More precisely, since ρ_l is small we have

$$\begin{array}{l} \int_{\mathbb{R}^n} \sum_{j,k=1}^n \left|\frac{\partial R^l_k}{\partial x_j}\right|^2(\tau,y)dy \leq \\ \\ \int_{\mathbb{R}^n} \sum_{j,k=1}^n \left(\left|\frac{\partial r^{l-1}_k}{\partial x_j}\right|^2 + \left|\frac{\partial v^{r,\rho,l-1}_k}{\partial x_j}\right|^2\right)(\tau,y)dy \end{array} \tag{437}$$

Hence, we get for generic $C^*_n > 0$

$$\begin{array}{l} \int_{\mathbb{R}^n} \sum_{j,k,m,p=1}^n \left|\frac{\partial r^l_k}{\partial x_j}\right|\left|\frac{\partial r^l_m}{\partial x_p}\right|(\tau,y)dy \leq \\ \\ C^*_n\left(\sum_{j,k=1}^n \left(\left|\frac{\partial v^{r,\rho,l-1}_k}{\partial x_j}\right|^2_0\right) C^2_\Gamma + \sum_{j,k=1}^n \left(\left|\frac{\partial r^{l-1}_k}{\partial x_j}\right|^2_0\right) C^2_\Gamma\right) + I_1 \\ \\ \leq C^*_n\left(C_r + (l-1)C_r + C^{l-1}_{1,2} + (l-1)C^{l-1}_{1,2}\right) + I_1 \end{array} \tag{438}$$

where

$$I_1=\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_k(l-1,y)\Gamma^l_{r,j}(\tau,x;s,y)dsdy$$

$$\times\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_m(l-1,y)\Gamma^l_{r,p}(\tau,x;s,y)dsdydx$$

$$+\int_{\mathbb{R}^n}\left(\sum_{j,k,m,p=1}^n\left(\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_k(l-1,y)\Gamma^l_{r,j}(\tau,x;s,y)dsdy\right)^2\right.\tag{439}$$

$$\left.+\left(\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_m(l-1,y)\Gamma^l_{r,p}(\tau,x;s,y)dsdy\right)^2\right)dx$$

and, similarly, for the first spatial derivatives we have

$$\begin{array}{l}\sum_{j,k,m,p,q=1}^n\left|\frac{\partial^2 r^l_k}{\partial x_j\partial x_q}\right|\left|\frac{\partial r^l_m}{\partial x_p}\right|(t,y)dy\leq\\ +\sum_{j,k,m,p,q=1}^n\left(\left|\frac{\partial^2 v^{r,\rho,l-1}_k}{\partial x_j\partial x_q}\right|_0^2+\left|\frac{\partial v^{r,\rho,l-1}_m}{\partial x_p}\right|_0^2\right)C^2_{\Gamma}\\ +\sum_{j,k,m,p,q=1}^n\left(\left|\frac{\partial^2 r^{l-1}_k}{\partial x_j\partial x_q}\right|_0^2+\left|\frac{\partial r^{l-1}_m}{\partial x_p}\right|_0^2\right)C^2_{\Gamma}+I_2\\ \leq C^*_n\left(C_r+(l-1)C_r\right)+I_2\end{array}\tag{440}$$

where

$$I_2=\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^n\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_{k,q}(l-1,y)\Gamma^{l,*}_{r,j}(\tau,x;s,y)dsdy$$

$$\times\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_{m,q}(l-1,y)\Gamma^{l,*}_{r,p}(\tau,x;s,y)dsdydx$$

$$+\int_{\mathbb{R}^n}\left(\sum_{j,k,m,p,q=1}^n\left(\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_{k,q}(l-1,y)\Gamma^l_{r,j}(\tau,x;s,y)dsdy\right)^2\right.\tag{441}$$

$$\left.+\left(\int_{l-1}^{\tau}\int_{\mathbb{R}^n}S^l_m(l-1,y)\Gamma^l_{r,p}(\tau,x;s,y)dsdy\right)^2\right)dx$$

Note that squared intergals S^l_i produce mixed terms such as products (of derivatives) of $v^{r,\rho,l}_i$ and r^l_i which can estimated by sums of squared terms again. This leads to estimates of I_1 and I_2 involving L^2-norms and L^4-norms in terms of functions of the previous time step. Note that all terms of S^l_i

have the factor ρ_l. For S^l_i we have

$$
\begin{array}{ll}
|S^l_i(l-1,.)|_{0,1} \leq \left|\rho_l \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial r^{l-1}_j}{\partial x_k}\right)(l-1,y)dy\right|_{0,1}\\
+\left|\rho_l \sum_{j=1}^n r^{l-1}_j\frac{\partial v^{r,\rho,l-1}_i}{\partial x_j}\right|_{0,1}+\left|\rho_l \sum_{j=1}^n v^{r,\rho,l-1}_j\frac{\partial r^{l-1}_i}{\partial x_j}\right|_{0,1}\\
+2\left|\rho_l \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_k}\right)(l-1,y)dy\right|_{0,1}\\
+\left|\rho_l \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v^{r,\rho,l-1}_m}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_m}\right)(l-1,y)dy\right|_{0,1}\\
\leq \rho_l C^*_n\left(C_r+lC_r+C^{l-1}_{1,2}+lC^{l-1}_{1,2}\right)
\end{array}
\tag{442}
$$

Similarly, using L^4- estimates -which we have from the previous time step inductively- we get

$$
|S^l_i(l-1,.)|^2_{0,1} \leq \rho_l C^*_n\left(C_r+lC_r+C^{l-1}_{1,2}+lC^{l-1}_{1,2}\right) \tag{443}
$$

Hence we have (generic C^*_n)

$$
I_1+I_2 \leq \rho_l C^*_n\left(C_r+lC_r+C^{l-1}_{1,2}+lC^{l-1}_{1,2}\right). \tag{444}
$$

Hence, we get for generic $C^*_n>0$

$$
\begin{array}{ll}
\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n\left|\frac{\partial r^l_k}{\partial x_j}\right|\left|\frac{\partial r^l_m}{\partial x_p}\right|(\tau,y)dy\\
\leq C^*_n\left(C_r+lC_r+C^{l-1}_{1,2}+lC^{l-1}_{1,2}\right).
\end{array}
\tag{445}
$$

The estimates

$$
\begin{array}{ll}
\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^n\left|\frac{\partial^2 r^l_k}{\partial x_j\partial x_q}\right|\left|\frac{\partial r^l_m}{\partial x_p}\right|(\tau,y)dy\\
\leq C^*_n\left(C_r+lC_r+C^{l-1}_{1,2}+lC^{l-1}_{1,2}\right).
\end{array}
\tag{446}
$$

and the related H^1- and H^2-estimates

$$
\begin{array}{ll}
\int_{\mathbb{R}^n}\sum_{j,k,m,p=1}^n\left(\left|\frac{\partial r^l_k}{\partial x_j}\right|^2+\left|\frac{\partial r^l_m}{\partial x_p}\right|^2\right)(\tau,y)dy\\
\leq C^*_n\left(C_r+lC_r+C^{l-1}_{1,2}+lC^{l-1}_{1,2}\right),
\end{array}
\tag{447}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\sum_{j,k,m,p,q=1}^n\left|\frac{\partial^2 r^l_k}{\partial x_j\partial x_q}\right|^2+\left|\frac{\partial r^l_m}{\partial x_p}\right|^2(\tau,y)dy\\
&\leq C^*_n\left(C_r+lC_r+C^{l-1}_{1,2}+lC^{l-1}_{1,2}\right).
\end{aligned}
\tag{448}
$$

are obtained similarly.

ad iii) We have defined the functions ϕ^l_i and r^l_i in terms of the functions $v^{r,\rho,l-1}_i$ and r^{l-1}_i. In the case $l=1$ we start with $v^{r,\rho,l-1}_i=h_i$, the initial data and with $r^0_i=r^1_i\equiv 0$. We have to estimate $v^{r,\rho,1}_i$ first in this special situation. Then we proceed with the case $l\geq 2$. In order to finish the first induction step $l=1$ we have to consider the growth of the local solution of $\mathbf{v}^{r,\rho,1}$. Note that for this first step this equals the local solution of the Navier-Stokes equation $\mathbf{v}^{\rho,l}$ because we have $\mathbf{r}^1\equiv 0$. Therefore at time step $l=1$ we may apply the machinery of the first step of this proof. Hence for $l=1$ we consider the functional series $\left(\mathbf{v}^{\rho,k,1}\right)_k$, where

$$
v^{\rho,k,1}_i=v^{\rho,0,1}_i+\sum_{m=1}^k\delta v^{\rho,m,1}_i. \tag{449}
$$

This series converges to the local solution $\mathbf{v}^{\rho,l}$ of the Navier-Stokes equation. Note that our symbolism allows for two alternative expressions for the members of this series. It can be denoted by $v^{r,\rho,k,1}_i=v^{\rho,k,l}_i+r^1_i=v^{\rho,k,l}_i$, or by the values of the map F_1 of the first step of this proof. Recall that the series (180) can be generated by an iterative application of the map

$$
F_1:\mathbf{f}\rightarrow\mathbf{v}^{f,\rho,l}, \tag{450}
$$

as described in step 1 above in case of general time step number l starting with the initial data $\mathbf{h}$. Then the function $F_1(\mathbf{h})=\mathbf{v}^{h,\rho,l}=\mathbf{v}^{\rho,0,1}$ satisfies the equation

$$
\left\lbrace
\begin{array}{ll}
\frac{\partial v^{h,\rho,1}_i}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v^{h,\rho,1}_i}{\partial x_j^2}+\rho_l\sum_{j=1}^n h_j\frac{\partial v^{h,\rho,1}_i}{\partial x_j}=\\
\\
\qquad\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial h_k}{\partial x_j}\frac{\partial h_j}{\partial x_k}\right)(\tau,y)dy,\ ,\\
\\
\mathbf{v}^{h,\rho,1}(0,.)=\mathbf{h}.
\end{array}
\right.
\tag{451}
$$

where $1\leq i\leq n$. Let Γ^1_0 be the fundamental solution of the equation

$$
\frac{\partial v^{h,\rho,1}_i}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v^{h,\rho,1}_i}{\partial x_j^2}+\rho_l\sum_{j=1}^n h_j\frac{\partial v^{h,\rho,1}_i}{\partial x_j}=0 \tag{452}
$$

Let C^0_h, C^1_h, and C^2_h be some constants such that

$$
|h_i|_0\leq C^0_h,\ |h_i|_1\leq C^1_h\ ,|h_i|_2\leq C^0_{1,2} \tag{453}
$$

for all $1 \leq i \leq n$. In terms of the fundamental solution Γ_0^0 of the equation (452) the components $v_i^{h,\rho,1}$ of the solution of (451) have the representation

$$v_i^{\rho,0,1}(\tau,x) = \int_{\mathbb{R}^n} h_i(y)\Gamma_0^1(\tau,x;0,y)dy$$

$$+\int_0^\tau \int_{\mathbb{R}^n} \rho_1 \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i}K_n(y-z)\right) \sum_{j,k=1}^n \left(\frac{\partial h_k}{\partial x_j}\frac{\partial h_j}{\partial x_k}\right)(s,z)\Gamma_0^1(\tau,x;s,y)dydzds. \tag{454}$$

From the maximum principle (cf. also Corollary 8.1.3. of [11]) we observe that

$$|\int_{\mathbb{R}^n} h_i(y)\Gamma_0^1(\tau,x;0,y)dy| \leq |h|_0 \tag{455}$$

for all $(\tau,x) \in [0,1]\times\mathbb{R}^n$. Hence, the solution $\mathbf{v}^{h,\rho,1} = \mathbf{v}^{\rho,0,1} = \mathbf{v}^{r,\rho,0,1}$ of equation (459) satisfies

$$|v_i^{h,\rho,1}|_0 \leq C^0 + |\rho_1 \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_i}K_n(x-y)\right) \sum_{j,k=1}^n \left(\frac{\partial h_k}{\partial x_j}\frac{\partial h_j}{\partial x_k}\right)(\tau,y)dy|C_\Gamma$$

$$\leq C_h^0 + \rho_1 C_\Gamma C_{1,2} \leq C_{1,2} + \rho_1 C_\Gamma C_{1,2} \tag{456}$$

where the definition of $C_{1,2}$ in terms of the function h is used. We are quite generous with the use of the bound $C_{1,2}$. As we described in the first step of this proof we may use the adjoint of the fundamental solution Γ_0^l and shift derivatives, we get the estimate

$$|v_i^{\rho,0,1}|_{1,2} \leq C_\Gamma^2 \left(C_{1,2}^0 + \rho_1 C_{1,2}\right) \tag{457}$$

Choosing

$$\rho_1 \leq \frac{1}{4C_\Gamma^2 C_{1,2}}, \tag{458}$$

we get

$$|v_i^{\rho,0,1}|_{1,2} \leq C_\Gamma^2 \left(C_{1,2}^0 + \tfrac{1}{4}\right) \tag{459}$$

Since $\mathbf{r}^1 \equiv 0$ we have convergence of the local scheme on $[0,1]\times\mathbb{R}^n$ where we choose $C_r = 0$ and $l = 1$ in the definition of ρ_l of the second step (or first step) of this proof. We may use ρ_1 with

$$\rho_1 \leq \frac{1}{C_n^* 4C_\Gamma^2 C_{1,2}}, \tag{460}$$

in order to get an estimate similar to (459) for the vector-valued function $\mathbf{v}$. Adding the estimate of the correction $\mathbf{v}^{\rho,1} - \mathbf{v}^{\rho,0,1} = \sum_{k\geq 1} |\delta v_i^k|_{1,2} \leq 1/4$ from the first step of this proof we get

$$|\mathbf{v}^{\rho,1}|_{1,2}^n = \sum_{i=1}^n |v_i^{\rho,1}|_{1,2} \leq C_\Gamma^2 \left(C_{1,2}^0 + 1\right) \tag{461}$$

where the constant $C_\Gamma \geq 1$ is from step 1 of this proof. Now we have to show that this constant $C_{1,2} = C_r$ is preserved for all $l \geq 2$. It is clear that

our definition of the constant C_r fits for the first step $l=1$, since we have chosen $\mathbf{r}^1\equiv 0$ in this first step. Indeed the constant C_r is determined in the induction step with respect to the time step number l. Furthermore we have determined ρ_1 for the first time step according to the convergence rule of the local scheme established in step 1 of this proof. Let us assume that $\mathbf{v}^{r,\rho,m}$ and $\mathbf{r}^m$ have been constructed for $m=1,\cdots,l-1$ as classical local solutions on the domains $[m-1,m]\times\mathbb{R}^n$. Furthermore assume that we have

$$|v_i^{r,\rho,m}|_0\leq C \quad\text{and}\quad |v_i^{r,\rho,m}|_{1,2}\leq C_{1,2} \tag{462}$$

for all $1\leq m\leq l-1$ and $1\leq i\leq n$. Furthermore, let us assume that

$$\begin{aligned}
&|v_i^{r,\rho,m}(\tau,.)|_{L^2}\leq C_n^*\left(C_{1,2}+mC_{1,2}\right)\\
&|v_i^{r,\rho,m}(\tau,.)|_{H^2}\leq C_n^*\left(C_{1,2}+mC_{1,2}\right)
\end{aligned}\tag{463}$$

for all $\tau\in[m-1,m]$.

Remark 2.23. Since we consider C_n^* in the definition of the constant $C_{1,2}$ to be generic we note that

$$|\mathbf{v}^{r,\rho,m}|^n_{1,2}\leq C_{1,2}. \tag{464}$$

Let us go back to the local Navier-Stokes equation on the domain $[l-1,l]\times\mathbb{R}^n$ which we wrote in the form (151). Let us repeat this equation here for the convenience of the reader. We have

$$\left\lbrace\begin{array}{ll}
\frac{\partial v_i^{r,\rho,l}}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v_i^{r,\rho,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l}\frac{\partial v_i^{r,\rho,l}}{\partial x_j}=\psi_i^l,\\
\\
\mathbf{v}^{r,\rho,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.),
\end{array}\right.\tag{465}$$

where

$$\begin{array}{ll}
\psi_i^l=r_{i,\tau}^l-\rho_l\nu\Delta r_i^l+\rho_l\sum_{j=1}^n r_j^l\frac{\partial r_i^l}{\partial x_j}\\
\\
-\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^l}{\partial x_j}\frac{\partial r_j^l}{\partial x_k}\right)(\tau,y)dy\\
\\
+\rho_l\sum_{j=1}^n r_j^l\frac{\partial v_i^{r,\rho,l}}{\partial x_j}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l}\frac{\partial r_i^l}{\partial x_j}\\
\\
-2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^l}{\partial x_j}\frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy\\
\\
+\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l}}{\partial x_j}\frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy.
\end{array}\tag{466}$$

Note that the right side of this equation involves the function $\mathbf{v}^{r,\rho,l}$ which is not known. However, the function $\mathbf{r}^l$ is known by our construction in

the previous substep, and determines together with the function $\mathbf{v}^{r,\rho,l-1}(l-1,.)$ the right side in the equation for the first approximation $\mathbf{v}^{r,\rho,0,l}$ of the function $\mathbf{v}^{r,\rho,0,l}$. We may estimate the growth of the function $\mathbf{v}^{r,\rho,0,l}$ first and then estimate the growth of the function $\mathbf{v}^{r,\rho,l}$ by estimating the difference

$$\mathbf{v}^{r,\rho,l}-\mathbf{v}^{r,\rho,0,l}=\sum_{k=1}^{\infty}\delta\mathbf{v}^{r,\rho,k,l} \tag{467}$$

which we know from the estimates of the previous steps of our proof. We have

$$\left\lbrace\begin{array}{ll}\frac{\partial v_i^{r,\rho,0,l}}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}(l-1,.)\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j}=\psi_i^{l,0}\\ \\ \mathbf{v}^{r,\rho,0,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.),\end{array}\right. \tag{468}$$

where

$$\begin{array}{ll}\psi_i^{l,0}=r^l_{i,\tau}-\rho_l\nu\Delta r_i^l+\rho_l\sum_{j=1}^n r_j^l\frac{\partial r_i^l}{\partial x_j}\\ \\ -\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial r^l_j}{\partial x_k}\right)(\tau,y)dy\\ \\ +\rho_l\sum_{j=1}^n r_j^l\frac{\partial v_i^{r,\rho,l-1}}{\partial x_j}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}\frac{\partial r_i^l}{\partial x_j}\\ \\ -2\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,y)dy\\ \\ +\rho_l\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(l-1,y)dy.\end{array} \tag{469}$$

We have determined $\mathbf{r}^l$ in the previous time step such that the first three

terms on the right side of (468) can be replaced. Indeed recall that we have

$$\begin{array}{l} r^l_{i,\tau}-\rho_l\nu\Delta r^l_i+\rho_l\sum_{j=1}^n r^{l-1}_j(l-1,.)\frac{\partial r^l_i}{\partial x_j}=\\ \\ +\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial r^{l-1}_j}{\partial x_k}\right)(l-1,y)dy\\ \\ -\rho_l\sum_{j=1}^n r^{l-1}_j\frac{\partial v^{r,\rho,l-1}_i}{\partial x_j}-\rho_l\sum_{j=1}^n v^{r,\rho,l-1}_j\frac{\partial r^{l-1}_i}{\partial x_j}\\ \\ +2\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_k}\right)(\tau,y)dy\\ \\ -\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,m=1}^n\left(\frac{\partial v^{r,\rho,l-1}_m}{\partial x_j}\frac{\partial v^{r,\rho,l-1}_j}{\partial x_m}\right)(l-1,y)dy+\phi^l_i. \end{array} \tag{470}$$

We may rewrite the function $\psi^{l,0}_i$ of (469), i.e., the right side of the first equation in (468) in the form

$$\psi^{l,0}_i=\phi^l_i+\left(\psi^{l,0}_i-\phi^l_i\right), \tag{471}$$

where

$$\begin{array}{l} \psi^{l,0}_i-\phi^l_i=\rho_l\sum_{j=1}^n(r^l_j-r^{l-1}_j)\frac{\partial r^l_i}{\partial x_j}\\ \\ -\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^l_k}{\partial x_j}\frac{\partial r^l_j}{\partial x_k}\right)(\tau,y)dy\\ \\ +\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r^{l-1}_k}{\partial x_j}\frac{\partial r^{l-1}_j}{\partial x_k}\right)(l-1,y)dy\\ \\ +\rho_l\sum_{j=1}^n(r^l_j-r^{l-1}_j)\frac{\partial v^{r,\rho,l-1}_i}{\partial x_j}+\rho_l\sum_{j=1}^n v^{r,\rho,l-1}_j\left(\frac{\partial r^l_i}{\partial x_j}-\frac{\partial r^{l-1}_i}{\partial x_j}\right)\\ \\ -2\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\left(\frac{\partial r^l_k}{\partial x_j}(\tau,y)-\frac{\partial r^{l-1}_k}{\partial x_j}(l-1,y)\right)\frac{\partial v^{r,\rho,l-1}_j}{\partial x_k}(l-1,y)\right)dy. \end{array} \tag{472}$$

Indeed, this difference of $\psi^{l,0}_i$ and ϕ^l_i may be obtained by subtracting equation (157) from equation (156) which we computed in (158) above. Note that all terms on the right side have the factor ρ_l. Estimating this difference boils down to estimating the difference $\mathbf{r}^l-\mathbf{r}^{l-1}$. In order to control the growth of the functions $v^{r,\rho,0,l}_i$ the estimate should be such that the functions ϕ^l_i dominate the differences $\psi^{l,0}_i-\phi^l_i$ in critcal areas where the moduli of the functions $v^{r,\rho,l-1}_i$ exceed a certain level. Summarizing we first estimate

$v_i^{r,\rho,0,l}$, where we rewrite (468) in the form

$$\left\lbrace\begin{array}{ll}\frac{\partial v_i^{r,\rho,0,l}}{\partial \tau}-\rho_l\nu\sum_{j=1}^n \frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}(l-1,.)\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j}=\phi_i^l+\left(\psi_i^{l,0}-\phi_i^l\right)\\ \\ \mathbf{v}^{r,\rho,0,l}(l-1,.)=\mathbf{v}^{r,\rho,l-1}(l-1,.),\end{array}\right. \tag{473}$$

and then we estimate $v_i^{r,\rho,l}-v_i^{r,\rho,0,l}$, where we use the second step of this proof, and the fact that the difference $\psi_i^l-\psi_i^{l,0}$ is given by

$$\begin{array}{ll}\psi_i^l-\psi_i^{l,0}=\\ \\ +\rho_l\sum_{j=1}^n r_j^l\left(\frac{\partial v_i^{r,\rho,l}}{\partial x_j}-\frac{\partial v_i^{r,\rho,l-1}}{\partial x_j}\right)+\rho_l\sum_{j=1}^n\left(v_j^{r,\rho,l}-v_j^{r,\rho,l-1}\right)\frac{\partial r_i^l}{\partial x_j}\\ \\ -2\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial r_k^l}{\partial x_j}\left(\frac{\partial v_j^{r,\rho,l}}{\partial x_k}-\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)\right)(\tau,y)dy\\ \\ +\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l}}{\partial x_j}\frac{\partial v_j^{r,\rho,l}}{\partial x_k}\right)(\tau,y)dy\\ \\ -\rho_l\int_{{\mathbb R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k^{r,\rho,l-1}}{\partial x_j}\frac{\partial v_j^{r,\rho,l-1}}{\partial x_k}\right)(\tau,y)dy.\end{array} \tag{474}$$

Indeed this is essentially estimated in the second step of this proof once the first difference is estimated.

Recall that

$$|r_i^l|_0\leq C_r^0,\ |r_i^l|_{1,2}\leq C_r^1,\ \text{and}\ |r_i^l|_{1,2}\leq C_r \tag{475}$$

We see that we have to estimate the difference $\delta r_i^l\equiv r_j^l-r_j^{l-1}(l-1,.)$ in order to get the estimate of that difference. Well, we an afford to do a rough estimate for this difference in the form

$$|\delta r_i^l|_{1,2}=|r_j^l-r_j^{l-1}(l-1,.)|_{1,2}\leq 2C_r. \tag{476}$$

Next we estimate the difference $\psi_i^{l,0}-\phi_i^l$. From equation (472) we get

$$|\psi_i^{l,0}-\phi_i^l|_{0,1}\leq\rho_l\left(4nC_r^2+4n^2C_KC_r^2+4nC_rC_{0,2}+8C_Kn^2C_rC_{0,2}\right)\ . \tag{477}$$

We get

$$|\psi_i^{l,0}-\phi_i^l|_{0,1}\leq\tfrac{1}{4}\ . \tag{478}$$

where we choose

$$\rho_l\leq\frac{1}{C_n^*C_r\left(C_r+lC_r\right)}. \tag{479}$$

Note the additional factor C_r and recall that C^*_n is a generic constant dependent on dimension n. Recall also that we have determined C_r in terms of the initial data function $\mathbf{h}$ above. Hence the scheme defined is global. We can now finish the proof. In order to control the growth of the functions $\mathbf{v}^{r,\rho,l}$ and the function $\mathbf{r}^l$ at time step l we fir $\mathbf{r}^l$ control 1) the growth of a linearized equation for $\mathbf{v}^{r,\rho,0,l}$, and 2) ensure that the correction $\mathbf{v}^{r\rho,l}-\mathbf{v}^{r,\rho,0,l}=\sum_{k=1}^{\infty}\delta\mathbf{v}^{r,\rho,k,0}$ is small enough by choosing ρ_l appropriately. Let us look at the linear approximation $\mathbf{v}^{r,\rho,0,l}$ of the function $\mathbf{v}^{r,\rho,l}$ first. From equation (171) we get the representation

$$\begin{array}{ll}
v_i^{r,\rho,0,l}(\tau,x)=\int_{\mathbb{R}^n}v^{r,\rho,l-1}(l-1,y)\Gamma^l_{v,0}(\tau,x;0,y)dy\\
\\
+\int_{l-1}^{\tau}\int_{\mathbb{R}^n}\psi_i^{l,0}(s,y)\Gamma^l_{v,0}(\tau,x;s,y)dyds
\end{array}\tag{480}$$

where $\Gamma^l_{v,0}$ is the fundamental solution of

$$\frac{\partial v_i^{r,\rho,0,l}}{\partial\tau}-\rho_l\nu\sum_{j=1}^n\frac{\partial^2 v_i^{r,\rho,0,l}}{\partial x_j^2}+\rho_l\sum_{j=1}^n v_j^{r,\rho,l-1}(l-1,.)\frac{\partial v_i^{r,\rho,0,l}}{\partial x_j}=0,\tag{481}$$

and $\psi_i^{l,0}$ is as in (171). We may rewrite

$$\begin{array}{ll}
v_i^{r,\rho,0,l}(\tau,x)=\int_{\mathbb{R}^n}v^{r,\rho,l-1}(l-1,y)\Gamma^l_{v,0}(\tau,x;0,y)dy\\
\\
+\int_{l-1}^{\tau}\int_{\mathbb{R}^n}\left(\left(\psi_i^{l,0}(s,y)-\phi_i^l(s,y)\right)+\phi_i^l(s,y)\right)\Gamma^l_{v,0}(\tau,x;s,y)dyds
\end{array}\tag{482}$$

where ϕ_i^l is constructed above and the difference $\psi_i^{l,0}-\phi_i^l$ is as defined in (472). Applying the maximum principle we may estimate the first term on the right side of equation (482) by the supremum (indeed, maximum) of the initial data. We have for all $(\tau,x)\in[l-1,l]\times\mathbb{R}^n$

$$\begin{array}{ll}
|v_i^{r,\rho,0,l}(\tau,x)|\leq\sup_{(\tau,x)\in[l-1,l]\times\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}v^{r,\rho,l-1}(l-1,y)\Gamma^l_{v,0}(\tau,x;0,y)dy\\
\\
+\int_{l-1}^{\tau}\int_{\mathbb{R}^n}\left(\frac{1}{4}+\phi_i^l(s,y)\right)\Gamma^l_{v,0}(\tau,x;s,y)dyds\Big|\leq\sup_{x\in\mathbb{R}^n}|v_i^{r,\rho,l-1}(l-1,x)|.
\end{array}\tag{483}$$

For this part the reasoning is the same as before in the case of the function $\mathbf{r}^l$. We use the maximum principle in order to estimate the term

$$\int_{\mathbb{R}^n}v^{r,\rho,l-1}(l-1,y)\Gamma^l_{v,0}(\tau,x;0,y)dy\tag{484}$$

and then consider the different cases making up the definition of ϕ_i^l together with the a priori estimate of the fundamental solution $\Gamma^l_{v,0}$ involved. Next we apply the second step of this proof which gives

$$\sum_{m=1}^{\infty}|\delta v_i^{r,\rho,k,l}|_{1,2}\leq\frac{1}{2}.\tag{485}$$

Note that

$$|v_i^{r,\rho,0,l}|_0 \leq |v_i^{r,\rho,l-1}| - \frac{1}{2}, \tag{486}$$

We conclude that

$$|v_i^{r,\rho,l}|_{1,2} \leq C_{1,2}. \tag{487}$$

The reasoning that

$$|v_i^{r,\rho,l}|_{H^2} \leq C_{1,2} + lC_{1,2} \tag{488}$$

reduces to the reasoning that

$$|v_i^{r,\rho,0,l}|_{H^2} \leq C_{1,2} + lC_{1,2} \tag{489}$$

The estimate (489) is obtained analogously to the H^2 estimate for r_i^l in the second substep of this third substep.

2.4 step 4: Global existence of classical solutions $\mathbf{v}^\rho$ and $\mathbf{v}$ resp.

The functions ϕ_i^l are bounded functions with supremum less or equal to 1 which vary with the time step number in general. At each time step $l \geq 1$ we defined them on the domain $(l-1,l]\times\mathbb{R}^n$. The functions $\mathbf{v}^{r,\rho}$ and $\mathbf{r}^l$ are only weakly differentiable with respect to time across the points $l = 0,1,2,\cdots$ in general. They are also Lipschitz. Furthermore, in the construction of each time step the functions ϕ_i^l are only Lipschitz with respect to the spatial variables in general. However, this is sufficient in order to find that the solution $\mathbf{v}^\rho = \mathbf{v}^{r,\rho} + \mathbf{r}$ constructed is classical. Consider the function $\mathbf{v}^{r,\rho,l}$ constructed at time step $l \geq 1$. The components $v_i^{r,\rho,l}$, $1 \leq i \leq n$ have the representation

$$v_i^{r,\rho,l} = v_i^{r,\rho,0,l} + \sum_{k=1}^{\infty} \delta v_i^{r,\rho,k,l}, \tag{490}$$

where $\delta v_i^{r,\rho,k,l} = v_i^{r,\rho,k,l} - v_i^{r,\rho,k-1,l}$ are determined successively by linear Cauchy problems with zero initial conditions. These linear Cauchy problems have bounded classical solutions with

$$\lim_{\tau\downarrow l-1} \frac{\partial \delta v^{r,\rho,k,l}}{\partial \tau}(\tau,x) = 0. \tag{491}$$

This implies that the regular behavior of the function $\mathbf{v}^{r,\rho,l}$ with respect to time is determined by the behavior of the function $\mathbf{v}^{r,\rho,0,l}$ as $\tau \downarrow l-1$ and the initial data (resp. the final data of the previous time step) $\mathbf{v}^{r,\rho,0,l}(l-1,.) = \mathbf{v}^{r,\rho,l-1}(l-1,.)$. We conclude that the function $\mathbf{v}^{r,\rho}$ is uniformly bounded continuous with respect to time. Furthermore it is Hölder continuous with respect to the spatial variables uniformly with respect to time. Similarly, the function $\tau \to \mathbf{r}(\tau,x)$ is only weakly differentiable at the integer values

$l \in \mathbb{N}$. Especially it is uniformly continuous with respect to the time variable τ and it is Hölder continuous with respect to the spatial variables uniformly in τ. Hence we conclude that the solution $\mathbf{v}^{r,\rho,l}$ on $[l-1,l] \times \mathbb{R}^n$ of the Navier-Stokes equation in transformed time coordinates τ, i.e., the function

$$\mathbf{v}^{\rho,l} = \mathbf{v}^{r,\rho,l} - \mathbf{r}^l \tag{492}$$

shares these properties. Hence, if we consider the family of fundamental solutions $\Gamma_v^{\rho,l}$ of the equations

$$\frac{\partial \Gamma_v^{\rho,l}}{\partial \tau} - \rho_l \nu \Delta \Gamma_v^{\rho,l} + \rho_l \sum_{j=1}^n v_j^{\rho,l} \frac{\partial \Gamma_v^{\rho,l}}{\partial x_j} = 0, \tag{493}$$

then we observe 1) that all exist in their Levy expansion form since the coefficient functions $v_j^{\rho,l}$ are uniformly continuous with respect to time and Hölder continuous with respect to the spatial variables 2) we can build a global bounded continuous coefficient functions $v_j^{\rho} : [0,\infty) \times \mathbb{R}^n$ out of the local coefficient functions $v_j^{\rho,l}$, where the local restrictions of v_j^{ρ} to the domain $[l-1,l] \times \mathbb{R}^n$ are equal to the functions $v_j^{\rho,l}$. The different coefficients ρ_l in the family of equations (493) are artefacts of the time transformations, of course. We can get rid of them by transforming back to original time coordinates

$$\tau \to t(\tau) := \rho_l \tau, \text{ if } \tau \in [l-1,l] \tag{494}$$

for all $l \geq 1$. Note that we can consider the function v_j^{ρ} as given because we have constructed it- the turn to the fundamental solution then gives us global classical solutions. Furthermore, the fundamental solutions Γ_v^l of

$$\frac{\partial \Gamma_v^l}{\partial t} - \nu \Delta \Gamma_v^l + \sum_{j=1}^n v_j^l \frac{\partial \Gamma_v^l}{\partial x_j} = 0 \tag{495}$$

exists on the transformed domains and in original coordinates, and we can build global bounded continuous coefficient functions $v_j : [0,\infty) \times \mathbb{R}^n$ out of the local coefficient functions v_j^l in original time coordinates, where the local restrictions of v_j to the domain $[l-1,l] \times \mathbb{R}^n$ are equal to the functions v_j^l. The related global fundamental solution can be obtained also by successive use of the the relation

$$\Gamma(t,x;s,y) = \int_{\mathbb{R}^n} \Gamma(t,x;s_1,z)\Gamma(s_1,z;s,y)dz \tag{496}$$

for $t > s_1 > s$, and where s_1 will run through the time step sizes $\sum_{m=1}^l \rho_m$ in orignal time coordinates. We conclude that the fundamental solution Γ_v of

$$\frac{\partial \Gamma_v}{\partial t} - \nu \Delta \Gamma_v + \sum_{j=1}^n v_j \frac{\partial \Gamma_v}{\partial x_j} = 0 \tag{497}$$

exists on $[0,T]\times\mathbb{R}^n$ for arbitrary $T>0$, where v_i is the function that equals v_i^l on the domain $[l-1,l]\times\mathbb{R}^n$ for all $l\geq 1$. Now for all $t\in[0,T]$ and $x\in\mathbb{R}^n$ we have the representation

$$\begin{array}{ll}v_i(t,x)=\int_{\mathbb{R}^n}h_i(y)\Gamma_v(t,x;0,y)dy+\int_0^t\int_{\mathbb{R}^n}\left(\frac{\partial v_m}{\partial x_l}\frac{\partial v_l}{\partial x_m}\right)(s,z)\times\\ \\ \times K_{,i}(z-y)\Gamma_v(t,x;s,y)dsdydz\end{array}\tag{498}$$

for $1\leq i\leq n$. Hence we have that $v_i\in C_b^{1,2}\left([0,T]\times\mathbb{R}^n\right)$ for all $T>0$. Furthermore the latter representation and the standard a priori estimates for classical fundamental solutions together with our assumptions on the initial data h_i show us that we have

$$v_i(t,.)\in H^2\tag{499}$$

for all $t\geq 0$, where we use Young's inequality. $\square$

3 Regularity, uniqueness, and extensions with external forces

The existence of global bounded classical solutions is essential in order to prove further regularity and uniqueness. Furthermore this is essential in order to extend the proof to equations with external forces, and to study asymptotic behavior. Maybe it is well known that bounded classical solutions lead to full regularity. However, we shall show this for the sake of completeness.

Next we prove regularity, i.e. we prove that for $1\leq i\leq n$ we have

$$v_i\in C^\infty\left([0,\infty)\times\mathbb{R}^n\right),\tag{500}$$

and for all $t\in[0,\infty)$ and $s\in\mathbb{R}$

$$v_i(t,.)\in H^s\left(\mathbb{R}^n\right).\tag{501}$$

One way to do this starts from the representation of the solution in (498) Note that we may write

$$v_i(t,x)=v_i^b(t,x)+p_{,i}(t,x),\tag{502}$$

where

$$v_i^b(t,x)=\int_{\mathbb{R}^n}h_i(y)\Gamma_v(t,x;0,y)dy\tag{503}$$

solves the Cauchy problem for the Burgers equation, i.e.,

$$\left\lbrace\begin{array}{l}\frac{\partial v_i^b}{\partial t}=\nu\sum_{j=1}^n\frac{\partial^2 v_i^b}{\partial x_j^2}-\sum_{j=1}^n v_j\frac{\partial v_i^b}{\partial x_j},\\ \\ \mathbf{v}(0,.)=\mathbf{h},\end{array}\right.\tag{504}$$

and $p_{,i}$ describes the gradient of the pressure, i.e., the solutions to the Poisson equations

$$-\Delta p_{,i} = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_k} v_j\right)\left(\frac{\partial}{\partial x_j} v_k\right) \tag{505}$$

for $1 \leq i \leq n$. Note that in terms of the Navier-Stokes solution function $\mathbf{v}$ the first equation in (504) is a linear parabolic equation for v_i^b for each $1 \leq i \leq n$. Hence a simple strategy for regularity is the following: start with some known regularity of $\mathbf{v}$ and consider (502). Then use regularity theory for linear parabolic equations in order to prove more regularity for the summand $\mathbf{v}^b$. Then starting with some regularity for the right side of (505) use elliptic regularity theory for Poission equations in order to get more regularity for $p_{,i}$. This gives more regularity for $\mathbf{v}$. Iteration of the process leads to full regularity. In a first step we look at the Poisson equation for p itself. Our main theorem tells us that we have

$$\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_k} v_j\right)\left(\frac{\partial}{\partial x_j} v_k\right) \in L^1. \tag{506}$$

(Well, it tells us that the right side is even in H^1). Hence the convolution with the fundamental solution n of the Poisson equation is a well defined locally integrable function, and we have

$$\begin{array}{l}\Delta\left(\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_k} v_j\right)\left(\frac{\partial}{\partial x_j} v_k\right) \star N\right)\\ = \sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_k} v_j\right)\left(\frac{\partial}{\partial x_j} v_k\right).\end{array} \tag{507}$$

Here recall that an L^1 estimate for the term in (506) can be reduced to an L^2 estimate. From our main result we know that for each time $t \in [0,\infty)$ the right side of (507) is in C^{α}. We get more regularity of this distributive solution for p from the standard result

Lemma 3.1. *Assume that $k \geq 0$, and Ω is an open set in $\mathbb{R}^n$. Assume u is a distribution solution of*

$$\Delta u = f \tag{508}$$

where the data $f \in C^{k+\alpha}(\Omega)$ along with $\alpha \in (0,1)$ and $k \geq 0$. Then $u \in C^{k+2+\alpha}(\Omega)$.

Hence we have $p(t,.) \in C^{2+\alpha}$ in the first step. Furthermore, we have

Lemma 3.2. *Suppose that for some $\alpha > 0$ we have $u \in C^{2+\alpha}$, where u satisfies the Poisson equation*

$$\Delta u = f \tag{509}$$

. Then for all $s > 0$ and $p \in (1,\infty)$ we have

$$f \in H^{s,p} \to u \in H^{2+s,p}. \tag{510}$$

This may be combined with Sobolev type embeddings for Zygmund spaces C^r_*, i.e.,

Lemma 3.3. *For $s > 0$ and $p \in (1, \infty)$ we have*

$$H^{s,p} \subset C^r_*, \text{ if } r = s - \frac{n}{p}. \tag{511}$$

This means that $p(t,.) \in H^{4,p} \subset C^{3+\alpha}$ where p may be large. From our main result we have $v_i(t,.) \in H^{2,p}$ for $p \geq 1$. Hence $v_i(t,.) \in C^{2+\alpha} \cap H^{4,p}$, and these are the first order coefficients of the equation for v^b_i. Then differentiating the equation (504) and using classical theory we get

$$v^b_i(t,.) \in C^{3,b} \cap H^{3,p} \tag{512}$$

for all $t \geq 0$ Hence, we have

$$v_i(t,.) \in C^{3,b} \cap H^{3,p} \tag{513}$$

for all $t \geq 0$ and $p > 1$. Iterating this argument we get full spatial regularity, i.e.,

$$v_i(t,.) \in C^\infty \text{ for all } \; t > 0. \tag{514}$$

The Navier-Stokes equation itself then tells us immediately some regularity of the first time derivative. We may then consider the time derivative of the Navier Stokes equation and consider this as a Navier-Stokes type equation for $\frac{\partial v_i}{\partial t}$ (with a certain source term and a potential term). We may iterate the argument above and get

$$\left(\frac{\partial}{\partial t} v_i\right)(t,.) \in C^\infty \text{ for all } \; t > 0. \tag{515}$$

Another way to prove regularity is the following which uses both the Leray projection form and the original form of the Navier-Stokes equation. We shall use the following We have $v_i \in C^{1,2}_b$ for all $1 \leq i \leq n$. Hence $v_{i,j}(t,.) \in C^1_b$, which implies for all $t \in [0, \infty)$ that

$$\sum_{j,k=1}^{n} \left(v_{j,k} v_{k,j}\right)(t,.) \in C^1_b \subset C^\alpha. \tag{516}$$

Next consider multiindices $\alpha = (\alpha_1, \cdots, , \alpha_n)$ and the multivariate partial derivative of order α, i.e.,

$$\frac{\partial^{|\alpha|}}{\partial x^{|\alpha|}} =: D^\alpha_x =:_{,\alpha} . \tag{517}$$

The spatial derivative of order α with $|\alpha|\geq 1$ of the Navier Stokes equation leads to

$$v_{i,\alpha,t}-\nu\Delta v_{i,\alpha}+\sum_{j=1}^{n}v_jv_{i,j,\alpha}=-p_{,i,\alpha}-\sum_{j=1}^{n}v_{j,\alpha}v_{i,j}-\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}v_{i,\beta}v_{i,j,\alpha-\beta}. \tag{518}$$

Next, assume inductively that

$$v_{i,\beta}\in C^{1,2}_b,\quad \text{for all } \beta<\alpha, \text{ and } \ 1\leq i\leq n, \tag{519}$$

and

$$v_{i,\beta}(t,.)\in L^2,\quad \text{for all } \beta<\alpha \text{ and all } t\in[0,\infty) \tag{520}$$

for all $\beta<\alpha$, and $1\leq i\leq n$. Then

$$v_{i,\alpha}(t,.)\in C^1_b \ \& \ \left(\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}v_{i,\beta}v_{i,j,\alpha-\beta}\right)(t,.)\in C^1_b\subset C^{\alpha}, \tag{521}$$

uniformly in the time variable t. Hence, we have the representation with fundamental solution Γ

$$\begin{array}{ll}v_{i,\alpha}(t,x)=\int_{\mathbb{R}^n}h_{i,\alpha}(y)\Gamma(t,x,0,y)dy+\\ \\ -\int_0^t p_{,i,\alpha}(s,y)\Gamma(t,x,s,y)dyds\\ \\ +\sum_{j=1}^{n}\left(v_{j,\alpha}v_{i,j}\right)(s,y)\Gamma(t,x,s,y)dsdy\\ \\ +\left(\sum_{0<\beta<\alpha}\binom{\alpha}{\beta}v_{i,\beta}v_{i,j,\alpha-\beta}\right)(s,y)\Gamma(t,x,s,y)dsdy,\end{array} \tag{522}$$

where Γ is the fundamental solution of

$$\frac{\partial\Gamma}{\partial t}-\nu\Delta\Gamma+\sum_{j=1}^{n}v_j\frac{\partial\Gamma}{\partial x_j}=0. \tag{523}$$

Hence, we have

$$v_{i,\alpha}\in C^{1,2}_b \tag{524}$$

Regularity with repect to time can be treated similarly using regularity with respect to the spatial variables.

Next we have uniqueness. Assume that $v_1,v_2\in C^{1,2}_b$ are two solutions of the Navier-Stokes equation. Then from the basic energy estimate

$$\sup_{0\leq t\leq T}|v_1-v_2|_0\leq|(v_1(0,.)-v_2(0,.))|_0\exp\left(\int_0^T|\nabla v_2|_{L^\infty}dt\right) \tag{525}$$

we get uniqueness. This is an application of Grönwall's lemma and can be found in standard texts. We mention it here for the sake of completeness.

Finally we mention extensions to equations with external forces and asymptotic behaviour. The solution of the Navier-Stokes equation wih external forces f_{ex} can be represented as

$$\begin{array}{ll} v_i(t,x)=\int_{\mathbb{R}^n}h_i(y)\Gamma(t,x,0,y)dy & \\ -\int_0^t\int_{\mathbb{R}^n}p_{,i}(s,y)\Gamma(t,x,s,y)dyds & (526)\\ +\int_0^t\int_{\mathbb{R}^n}f_{ex}(s,y)\Gamma(t,x,s,y)dsdy. & \end{array}$$

Since the gradient of the pressure is bounded in case $f_{ex}\equiv 0$ we see that the global solution converges to 0 as $t\uparrow\infty$.

4 The algorithm

The construction of the solution above can be extended to boundary value problems straightforwardly. We consider first initial-boundary value problems and second initial-boundary value problems. The observation is that the global existence for these problems boils down to the existence of related scalar first initial-boundary value problems and second initial-boundary value problems of parabolic type. Furthermore, the scheme proposed has linear subproblems of parabolic type which can be computed explicitly. Expansions of this form are considered in [8]. Note that there is a difference to the Taylor expansion (operator form) which applies only for affine coefficients in general (cf. [1] and [7]). Furthermore the subproblems considered here have been implemented in the context of weighted Monte-Carlo methods in finance (cf. [10], [5], and [4]). The discussion here is rather conceptual. Further details of implementation and error estimates will be considered in subsequent paper.

Since we have proved that the solution is bounded we may set up a uniform time discretization scheme. The size of the time steps is limited by a time step size which ensures the convergence of the local time scheme. A lower bound of this time step size can be extracted the global existence proof. The step size numbers ρ_l may be increased as time goes by if there are smoothing effects due to the strictly parabolic subproblems.

Since we have proved that the solution $v_i,\ 1\leq i\leq n$ of the incompressible Navier-Stokes equation is globaly bounded and Hölder continuous with respect to the spatial variables uniformly in time, we know that the fundamental solution Γ_v of the equation

$$\frac{\partial\Gamma_v}{\partial t}-\nu\Delta\Gamma_v+\sum_j v_j\frac{\partial\Gamma_v}{\partial x_j}=0 \tag{527}$$

exists (given $\mathbf{v}$). Hence, in terms of the solution itself the solution of the

Cauchy problem has the representation (original time coordinates)

$$\begin{array}{l} v_i(t,x) = \int_{\mathbb{R}^n} h_i(y)\Gamma_v(t,x;0,y)dy+ \\ \int_{l-1}^{l}\int_{\mathbb{R}^n}\sum_{m,l=1}^{n}\left(v_{l,m}v_{m,l}\right)(s,z)K(y-z)\Gamma_v(t,x;s,y)dy. \end{array} \tag{528}$$

This representation cannot be used for computation of course, since we do not now the solution. However, we have shown that for a time step size $\rho<1$ which is small enough we may compute the solution in a time -discretized scheme where at each time step l we compute successive approximations $v^{\rho,k,l}$ with initial values from the previous time step.

Note that the precise values from the the previous time step are $v^{\rho,l-1}$ which we do not know except for the case $l=1$ where $v^{\rho,l-1}_i=h_i$, i.e. equal the original initial data (well even these have to be approximated upon implementation). Hence, the initial data of the previous time step are some given by a function $v^{\rho,l-1,*}$ which approximates $v^{\rho,l-1}_i$. The approximation of $v^{\rho,l}_i$ is then computed by iterative approximations $v^{\rho,k,l,*}_i$ for $k\geq 0$ where $v^{\rho,0,l,*}_i$ solves

$$\left\lbrace\begin{array}{l} \frac{\partial v^{\rho,0,l,*}_i}{\partial\tau}-\rho\nu\sum_{j=1}^{n}\frac{\partial^2 v^{\rho,0,l,*}_i}{\partial x_j^2}+\rho\sum_{j=1}^{n}v^{\rho,l-1,*}_j\frac{\partial v^{\rho,0,l,*}_i}{\partial x_j}=\\ \\ \rho\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^{n}\left(\frac{\partial v^{\rho,l-1,*}_k}{\partial x_j}\frac{\partial v^{\rho,l-1,*}_j}{\partial x_k}\right)(\tau,y)dy,\\ \\ \mathbf{v}^{\rho,0,l,*}(l-1,.)=\mathbf{v}^{\rho,l-1,*}(l-1,.), \end{array}\right. \tag{529}$$

and for $k\geq 1$ recursively defined functions $\mathbf{v}^{\rho,k,l,*}$ are determined by the respective solutions of

$$\left\lbrace\begin{array}{l} \frac{\partial v^{\rho,k,l,*}_i}{\partial\tau}-\rho\nu\sum_{j=1}^{n}\frac{\partial^2 v^{\rho,k,l,*}_i}{\partial x_j^2}+\rho\sum_{j=1}^{n}v^{\rho,k-1,l,*}_j\frac{\partial v^{\rho,k,l,*}_i}{\partial x_j}=\\ \\ \rho\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^{n}\left(\frac{\partial v^{\rho,k-1,l,*}_k}{\partial x_j}\frac{\partial v^{\rho,k-1,l,*}_j}{\partial x_k}\right)(\tau,y)dy,\\ \\ \mathbf{v}^{\rho,k,l,*}(l-1,.)=\mathbf{v}^{\rho,l-1,*}(l-1,.). \end{array}\right. \tag{530}$$

We have to stop after finitely many steps. Hence we shall have

$$v^{\rho,l,*}_i=v^{\rho,m,l,*}_i \tag{531}$$

for some m if we perform iterations $k=0,\cdots,m$ at time step l. Let $\Gamma^{l,0}_*$ be the fundamental solution of the equation

$$\frac{\partial\Gamma^{l,0}_*}{\partial\tau}-\nu\Delta\Gamma^{l,0}_*+\sum_j v^{\rho,l-1,*}_j\frac{\partial\Gamma^{l,0}_*}{\partial x_j}=0. \tag{532}$$

Then the solution $v_i^{\rho,l,0,*}$ of (529) has the representation

$$v_i^{\rho,l,0,*}(t,x) = \int_{\mathbb{R}^n} v_i^{\rho,l-1,*}(l-1,y)\Gamma_*^{l,0}(\tau,x;l-1,y)dy+$$

$$\int_{l-1}^{\tau}\int_{\mathbb{R}^n}\sum_{m,l=1}^n \left(v_{l,m}^{\rho,k-1,l,*}v_{m,l}^{\rho,k-1,l,*}\right)(s,z)K_{,i}(y-z)\Gamma_*^{l,0}(\tau,x;s,y)dy. \tag{533}$$

Furthermore, let $\Gamma_*^{l,k}$ be the fundamental solution of the equation

$$\frac{\partial \Gamma_*^{l,k}}{\partial \tau} - \nu\Delta\Gamma_*^{l,k} + \sum_j v_j^{\rho,k-1,l,*}\frac{\partial \Gamma_*^{l,k}}{\partial x_j} = 0. \tag{534}$$

Then the solution (530) has the representation

$$v_i^{\rho,k,l,*}(t,x) = \int_{\mathbb{R}^n} v_i^{\rho,l-1,*}(y)\Gamma_*^{l,k}(\tau,x;0,y)dy+$$

$$\int_{l-1}^{\tau}\int_{\mathbb{R}^n}\sum_{m,l=1}^n \left(v_{l,m}^{\rho,k-1,l,*}v_{m,l}^{\rho,k-1,l,*}\right)(s,z)K_{,i}(y-z)\Gamma_*^{l,k}(\tau,x;s,y)dy. \tag{535}$$

This scheme involves the fundamental solutions of equations of type

$$\frac{\partial u}{\partial t} = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \tag{536}$$

essentially. For computational point of view it is interesting that the solution has the locally pointwise valid representation

$$p(t,x,0,y) = \frac{1}{\sqrt{4\pi t}^n}\exp\left(-\frac{\sum_{i=1}^n \Delta x_i^2}{4t}\right)\left(\sum_{k=0}^{\infty} d_k(t,x,y)t^k\right), \tag{537}$$

for $j = 1,\cdots,n$, i.e., the representation is valid on some time interval. However, this fits with our scheme since this is defined locally in time anyway. Note that we have the coefficients outside the exponential. This implies that the first term is damping the polynomial terms d_k as the moduli of the $\Delta x_i = (x_i - y_i)$ become large. The coefficient functions d_k have explict representations in terms of the coefficient functions b_i: for $k = 0$ we have

$$d_0(t,x,y) = \exp\left(\sum_m (y_m - x_m)\int_0^1 b_m(t, y+s(x-y))ds\right), \tag{538}$$

$$d_m(t,x,y) = \sum_{k=1}^m \frac{k}{m} d_{m-k}\int_0^1 R_{k-1}(t,y+s(x-y),y)s^k ds \tag{539}$$

with

$$\begin{array}{rl} R_{k-1}(t,x,y) = & \frac{\partial}{\partial t}c_{k-1} + \Delta c_{k-1} + \sum_{l=1}^n\sum_{r=0}^{k-1}\left(\frac{\partial}{\partial x_l}c_r\frac{\partial}{\partial x_l}c_{k-1-r}\right) \\ & +\sum_i b_i(x)\frac{\partial}{\partial x_i}c_{k-1} \end{array} \tag{540}$$

If the coefficients b_i are given in terms of bounded analytical expansions (finite Fourier series for example), then the functions d_k can be computed explicitly (cf. [8]). Note that we cannot use Taylor expansions (operator form) as in [1] since complete sets of analytic vectors are difficult to define if the coefficient functions are not affine (which is true in case of the Navier-Stokes equation). These analytical expansions have been proved to be quite efficient in a different context (cf. [4, 5, 10, 9]). Fluid dynamical models in applied sciences have boundaries of course, so let have a look how our algorithm can be adapted to these situations. We consider boundary value problems which are related to the second initial-boundary value problem for scalar parabolic equations - cf. [3] for a classical treatment. We consider problems on the domain $[0,T]\times\Omega$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain. Let $B:=\{0\}\times\Omega$, $B_T=\{0\}\times\Omega$, and let $S=\partial\Omega\setminus(B\cup B_T)$. Then we consider the following initial-boundary value problem on $[0,T]\times\Omega$. Let $\alpha_i:[0,T]\times\Omega\rightarrow\mathbb{R}$, and $g_i:[0,T]\times\Omega\rightarrow\mathbb{R}$ be $2n$ functions. We consider a problem for $v_i,\ 1\leq i\leq n$, where

$$\begin{cases}\frac{\partial v_i}{\partial t}-\nu\sum_{j=1}^n\frac{\partial^2 v_i}{\partial x_j^2}+\sum_{j=1}^n v_j\frac{\partial v_i}{\partial x_j}=\\ \\ \qquad\int\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\sum_{j,k=1}^n\left(\frac{\partial v_k}{\partial x_j}\frac{\partial v_j}{\partial x_k}\right)(t,y)dy,\\ \\ \frac{\partial}{\partial\nu}v_i(t,x)+\alpha_i(t,x)v_i(t,x)=g_i(t,x)\text{ on }[0,T]\times S,\\ \\ \mathbf{v}(0,.)=\mathbf{h}.\end{cases}\tag{541}$$

The scheme we proposed for the Cauchy problem can be adapted to this situation straightforwardly. Consider a time discretization in transformed coordinates and assume thhat $\mathbf{v}^{\rho,l-1}$ has computed for $l-1$, where $l\geq 0$ If $l=0$ we set $\mathbf{v}^{\rho,-1}=\mathbf{h}$ which is known. We choose a fixed ρ independent of the time step number l. For $l\geq 1$ we have functions $\mathbf{v}^{\rho,l}:[l-1,l]\times\Omega\rightarrow\mathbf{R}^n$ on successive domains where the final data of the function $\mathbf{v}^{\rho,l-1}$ are the initial dats of the function $\mathbf{v}^{\rho,l}$. Each function $\mathbf{v}^{\rho,l}$ is determined as a limit of a functional series $\left(\mathbf{v}^{\rho,k,l}\right)_k$. Note that transformation to coordinates $t=\rho\tau$ leads to a domain $[0,RT]\times\Omega$, where $R=\frac{1}{\rho}$. Having computed the $k-1$ iteration step of the lth time step the problem for $\mathbf{v}^{\rho,k,l}$ is given by n linear parabolic scalr problem which are classical initial-boundary value

problems of second type.

$$\left\lbrace\begin{array}{ll}
\frac{\partial v_i^{\rho,k,l}}{\partial\tau}-\rho\nu\sum_{j=1}^n\frac{\partial^2 v_i^{\rho,k,l}}{\partial x_j^2}+\rho\sum_{j=1}^n v_j^{\rho,k-1,l}\frac{\partial v_i^{\rho,k,l}}{\partial x_j}=\\
\\
\qquad\rho\sum_{j,m=1}^n\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_i}K_n(x-y)\right)\left(\frac{\partial v_m^{\rho,k-1,l}}{\partial x_j}\frac{\partial v_j^{\rho,k-1,l}}{\partial x_m}\right)(\tau,y)dy,\\
\\
\frac{\partial}{\partial\nu}v_i^{\rho,k,l}(\tau,x)+\alpha_i(t,x)v_i^{\rho,k,l}(\tau,x)=g_i(t,x)\text{ on }[0,RT]\times S,\\
\\
\mathbf{v}^{\rho,k,l}(l-1,.)=\mathbf{v}^{\rho,l-1}(l-1,.),
\end{array}\right.\tag{542}$$

where for $k=0$ we set $\mathbf{v}^{\rho,k-1,l}=\mathbf{v}^{\rho,-1,l}=\mathbf{v}^{\rho,l-1}(l-1,.)$ in order to determine the first order coefficients in the first step of a local iteration. We could add external forces in (542) but we leave it out for simplicity. The local convergence of the scheme (with the right choice of ρ is proved similarly as in the first step of the proof of the main theorem above, i.e., by proving the convergence of the functional series in the form

$$\mathbf{v}^{\rho,l}=\mathbf{v}^{\rho,0,l}+\sum_{k=1}^{\infty}\delta\mathbf{v}^{\rho,k,l},\tag{543}$$

where

$$\delta\mathbf{v}^{\rho,k,l}\downarrow 0\text{ as}\downarrow 0.\tag{544}$$

Note that the initial conditions and the boundary conditions for the functions $\delta v_i^{\rho,k,l}$ simplify to

$$\begin{array}{ll}
\frac{\partial}{\partial\nu}\delta v_i^{\rho,k,l}(\tau,x)+\alpha_i(t,x)\delta v_i^{\rho,k,l}(\tau,x)=0\text{ on }[0,RT]\times S,\\
\\
\mathbf{v}^{\rho,k,l}(l-1,.)=0.
\end{array}\tag{545}$$

This leads to representations of the solution in terms of the fundamental solutions Γ_k^l of the equations

$$\frac{\partial v_i^{\rho,k,l}}{\partial\tau}-\rho\nu\sum_{j=1}^n\frac{\partial^2 v_i^{\rho,k,l}}{\partial x_j^2}+\rho\sum_{j=1}^n v_j^{\rho,k-1,l}\frac{\partial v_i^{\rho,k,l}}{\partial x_j}=0.\tag{546}$$

Let us start with the representation for $\mathbf{v}^{\rho,0,l}$. The solution is given in the form

$$\begin{array}{ll}
v_i^{\rho,0,l}(\tau,x)=\int_\Omega v_i^{\rho,l-1}(l-1,y)\Gamma_0^l(\tau,y;0,y)dy\\
\\
+\int_{l-1}^\tau\int_\Omega\rho\sum_{j,m=1}^n\int_\Omega\left(\frac{\partial}{\partial x_i}K_n(y-z)\right)\left(\frac{\partial v_m^{\rho,l-}}{\partial x_j}\frac{\partial v_j^{\rho,l-1}}{\partial x_m}\right)(l-1,z)dz\Gamma_0^l(\tau,x;s,y)dsdy\\
\\
+\int_{l-1}^\tau\int_S\phi_i(s,y)\Gamma_0^l(\tau,x;s,y)dsdy.
\end{array}\tag{547}$$

where ϕ_i is the solution of the integral equation

$$\frac{1}{2}\phi_i(\tau,x)=\int_{l-1}^{\tau}\int_S K_\Gamma(\tau,x;s,y)\phi_i(s,y)dsdy+f_i(\tau,x) \tag{548}$$

along with the kernel

$$K_\Gamma(\tau,x;s,y)=\frac{\partial}{\partial\nu}\Gamma_0^l(\tau,x;s,y)+\alpha_i(\tau,x)\Gamma_0^l(\tau,x;s,y), \tag{549}$$

and the functions f_i which satisfy

$$\begin{array}{ll}f_i(\tau,x)=\int_\Omega K_\Gamma(\tau,x;0,y)v_i^{\rho,l-1}(l-1,y)dy-g_i(\tau,x)\\ -\rho\int_\Omega\int_{l-1}^{\tau}\int_\Omega K_\Gamma(\tau,x;s,y)\\ \times\sum_{j,m=1}^n\int_\Omega\left(\frac{\partial}{\partial x_i}K_n(y-z)\right)\left(\frac{\partial v_m^{\rho,l-1}}{\partial x_j}\frac{\partial v_j^{\rho,l-1}}{\partial x_m}\right)(l-1,z)dzdy\end{array} \tag{550}$$

Well for the corrections $\delta v_i^{\rho,k,l}$ these expressions become simplified. We

$$\begin{array}{ll}\delta v_i^{\rho,k,l}(\tau,x)=\\ +\int_{l-1}^{\tau}\int_\Omega\rho\sum_{j,m=1}^n\int_\Omega\left(\frac{\partial}{\partial x_i}K_n(y-z)\right)\left(\frac{\partial v_m^{\rho,k-1,l}}{\partial x_j}\frac{\partial v_j^{\rho,k-1,l}}{\partial x_m}\right)(s,z)dz\Gamma_k^l(\tau,x;s,y)dsdy\\ +\int_{l-1}^{\tau}\int_S\phi_i^k(s,y)\Gamma_k^l(\tau,x;s,y)dsdy.\end{array} \tag{551}$$

where ϕ_i^k is the solution of the integral equation

$$\frac{1}{2}\phi_i^k(\tau,x)=\int_{l-1}^{\tau}\int_S K_\Gamma^k(\tau,x;s,y)\phi_i^k(s,y)dsdy+f_i^k(\tau,x) \tag{552}$$

along with the kernel

$$K_\Gamma^k(\tau,x;s,y)=\frac{\partial}{\partial\nu}\Gamma_k^l(\tau,x;s,y)+\alpha_i(\tau,x)\Gamma_k^l(\tau,x;s,y), \tag{553}$$

and the functions f_i^k which satisfy

$$\begin{array}{ll}f_i^k(\tau,x)=\\ -\rho\int_\Omega\int_{l-1}^{\tau}\int_\Omega K_\Gamma(\tau,x;s,y)\\ \times\sum_{j,m=1}^n\int_\Omega\left(\frac{\partial}{\partial x_i}K_n(y-z)\right)\left(\frac{\partial v_m^{\rho,k-1,l}}{\partial x_j}\frac{\partial v_j^{\rho,k-1,l}}{\partial x_m}\right)(s,z)dzdyds.\end{array} \tag{554}$$

Well, the functions ϕ_i and ϕ_i^k have an explicit Levy-type expansion. For $k=0$ we have it in the form

$$\begin{array}{l}\frac{1}{2}\phi_i(\tau,x)=\\ \\ f_i(\tau,x)+\sum_{m=1}^{\infty}\int_{l-1}^{\tau}\int_S K_{\Gamma}^m(\tau,x;s,y)f_i(s,y)dsdy,\end{array}\tag{555}$$

where

$$K_{\Gamma}^1(\tau,x;s,y)=K_{\Gamma}(\tau,x;s,y),\tag{556}$$

and

$$K_{\Gamma}^{m+1}(\tau,x;s,y)=\int_{l-1}^{\tau}\int_{\Omega}K_{\Gamma}^1(\tau,x;\sigma,z)K_{\Gamma}^m(\sigma,z;s,y)dyds.\tag{557}$$

Similarly, for $k>0$ we have it in the form

$$\begin{array}{l}\frac{1}{2}\phi_i^k(\tau,x)=\\ \\ f_i^k(\tau,x)+\sum_{m=1}^{\infty}\int_{l-1}^{\tau}\int_S K_{\Gamma}^{m,k}(\tau,x;s,y)f_i^k(s,y)dsdy,\end{array}\tag{558}$$

where

$$K_{\Gamma}^{1,k}(\tau,x;s,y)=K_{\Gamma}^k(\tau,x;s,y),\tag{559}$$

and

$$K_{\Gamma}^{m+1,k}(\tau,x;s,y)=\int_{l-1}^{\tau}\int_{\Omega}K_{\Gamma}^{1,k}(\tau,x;\sigma,z)K_{\Gamma}^{m,k}(\sigma,z;s,y)dyds.\tag{560}$$

It is possible to extend this work to Navier-Stokes on manifolds. Another interesting problem is the extension to equations where the coefficients satisfy the Hörmander conditions and may have a stochastic force term. These problems will be studied in a subsequent paper.

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